Stanford QIQC Seminar

Springs, Symmetries, and Speedups

Efficient quantum circuits for irreps of SU(n) and Ramanujan quantum expanders

Sid Jain

Joint work with Vishnu Iyer, Rolando Somma, and Stephen Jordan

PART 01

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Symmetries

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Implementing the irreps of $SU(n)$

The high-dimensional SU(2) problem

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Naive Hamiltonian simulation costs $\mathrm{poly}(N)$ because $||J|| = \Omega(N)$

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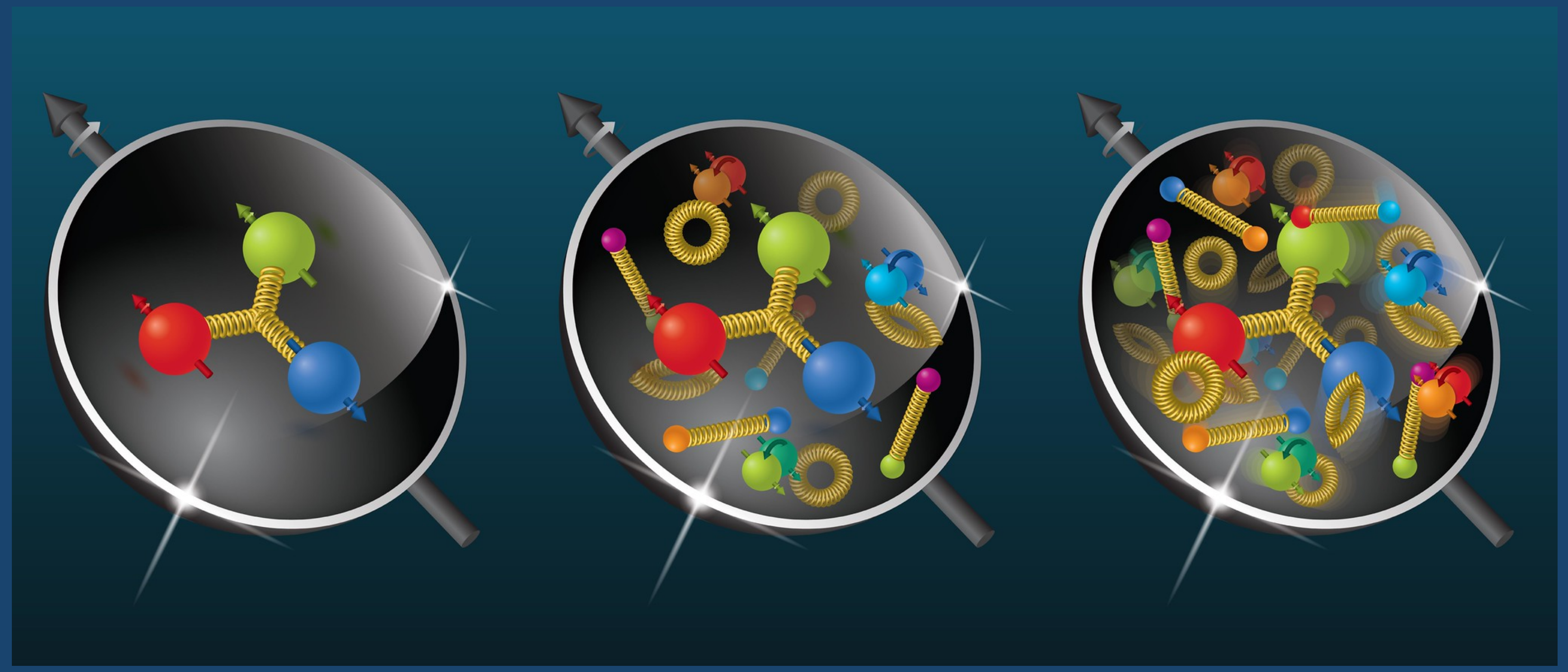

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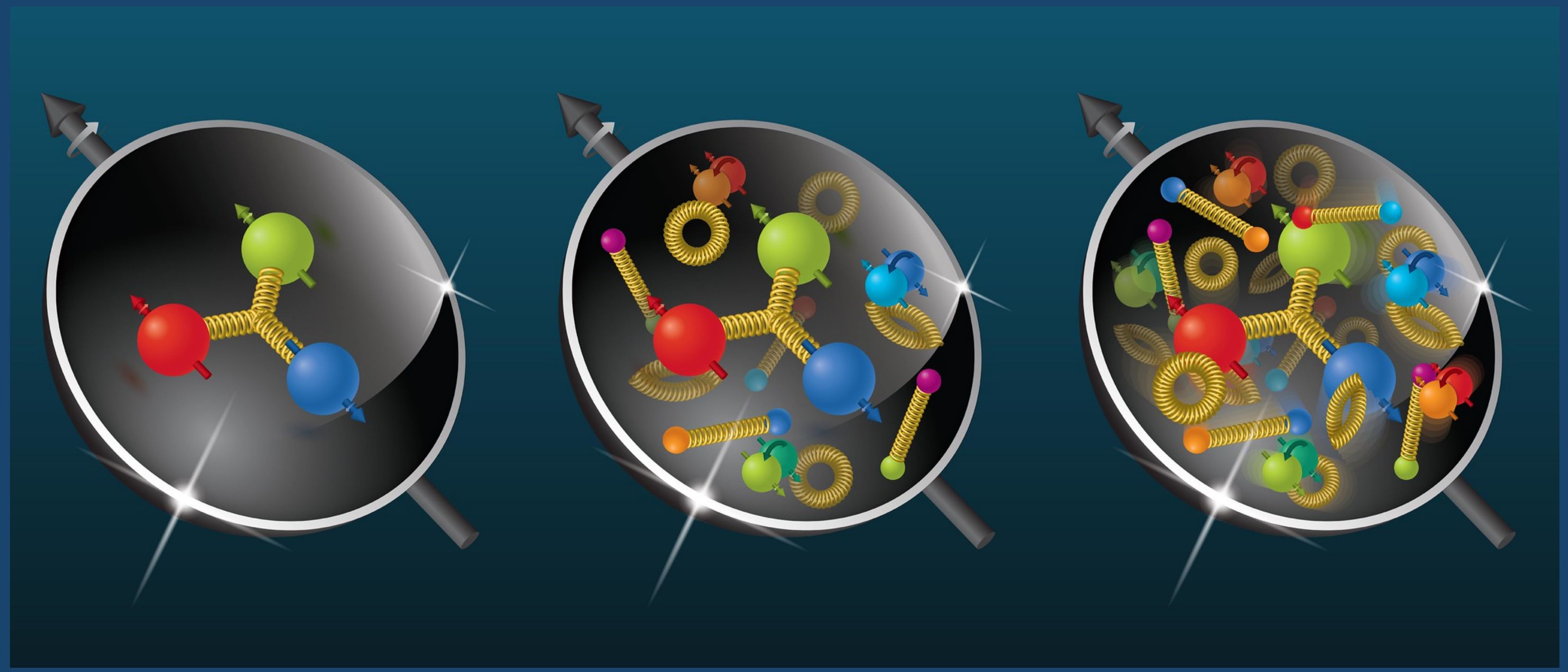

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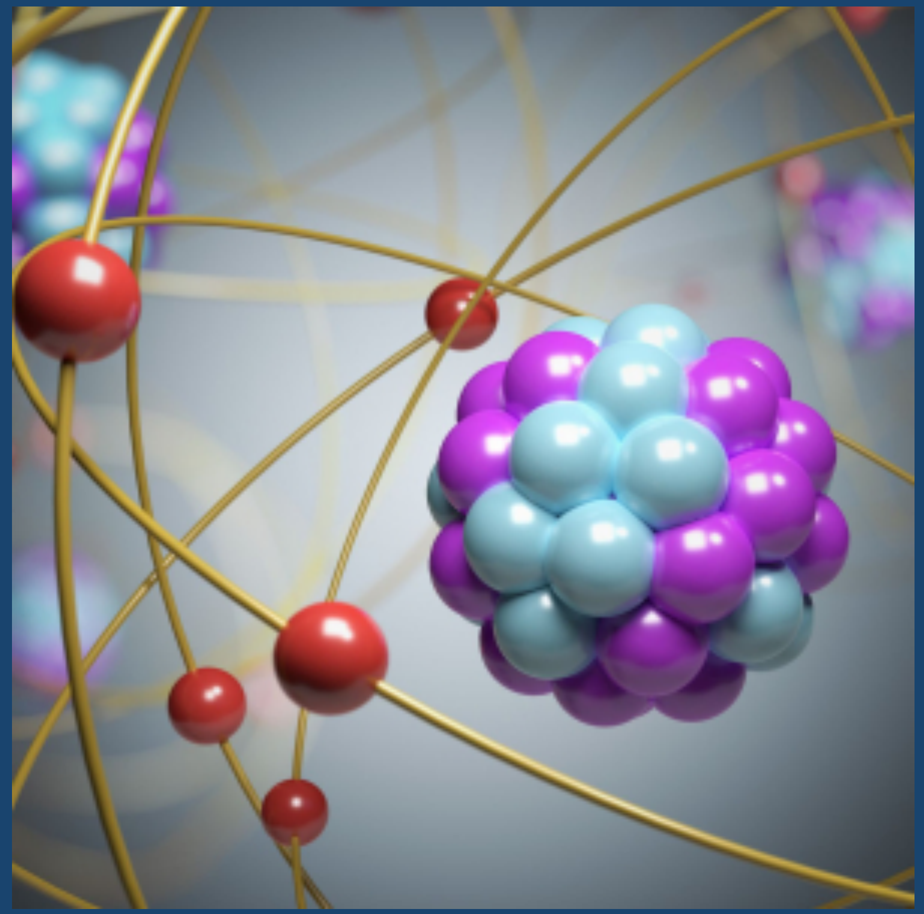

Nuclear Physics

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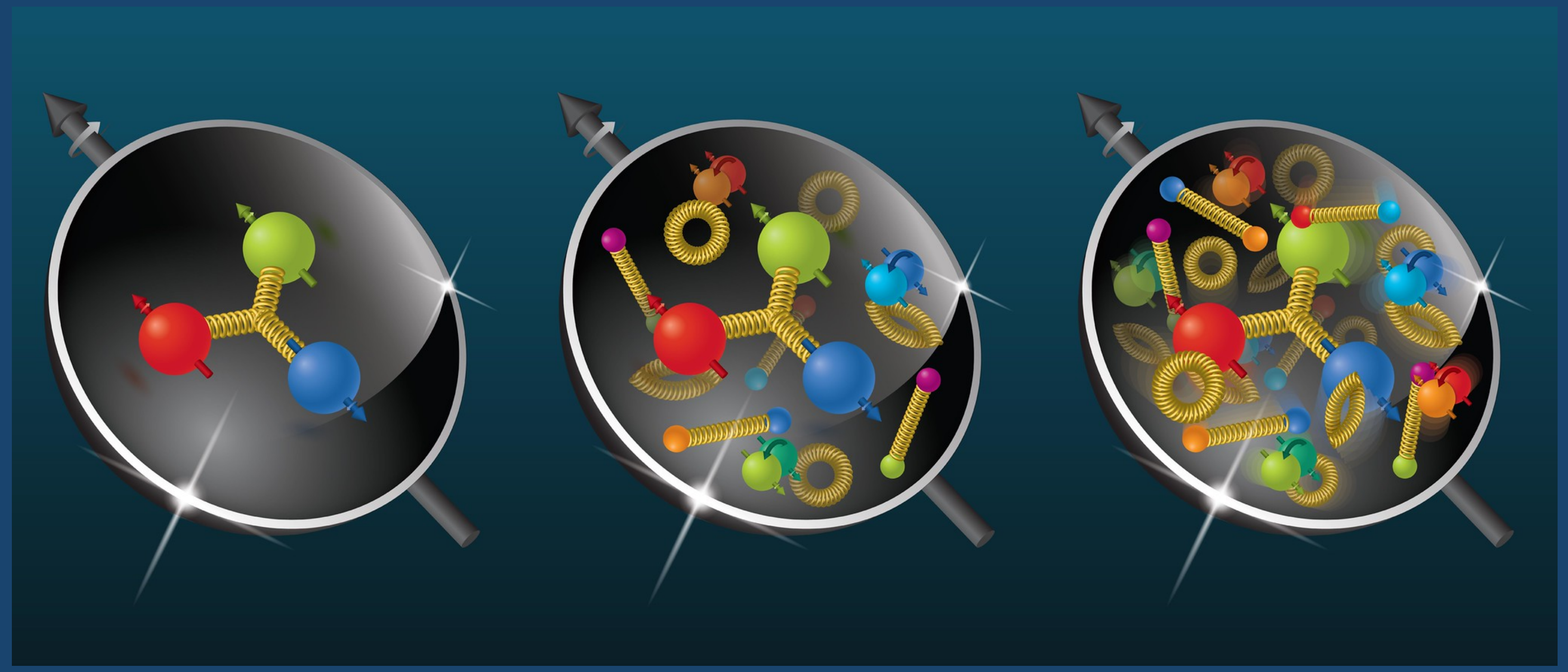

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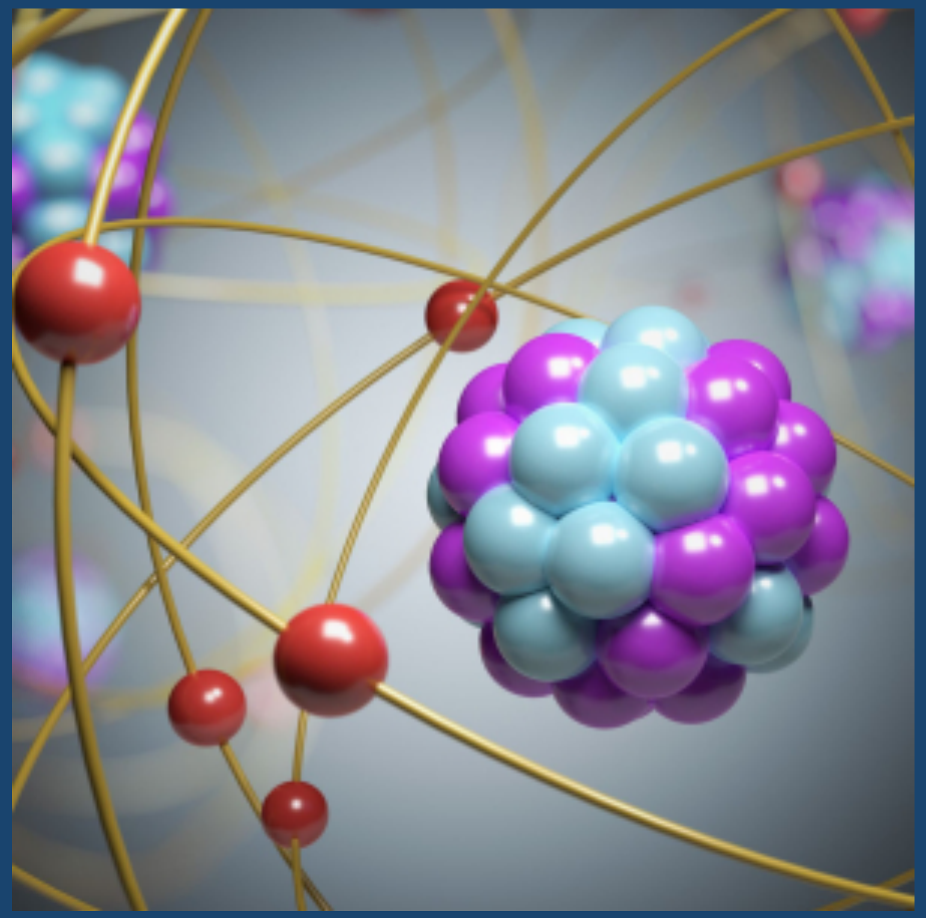

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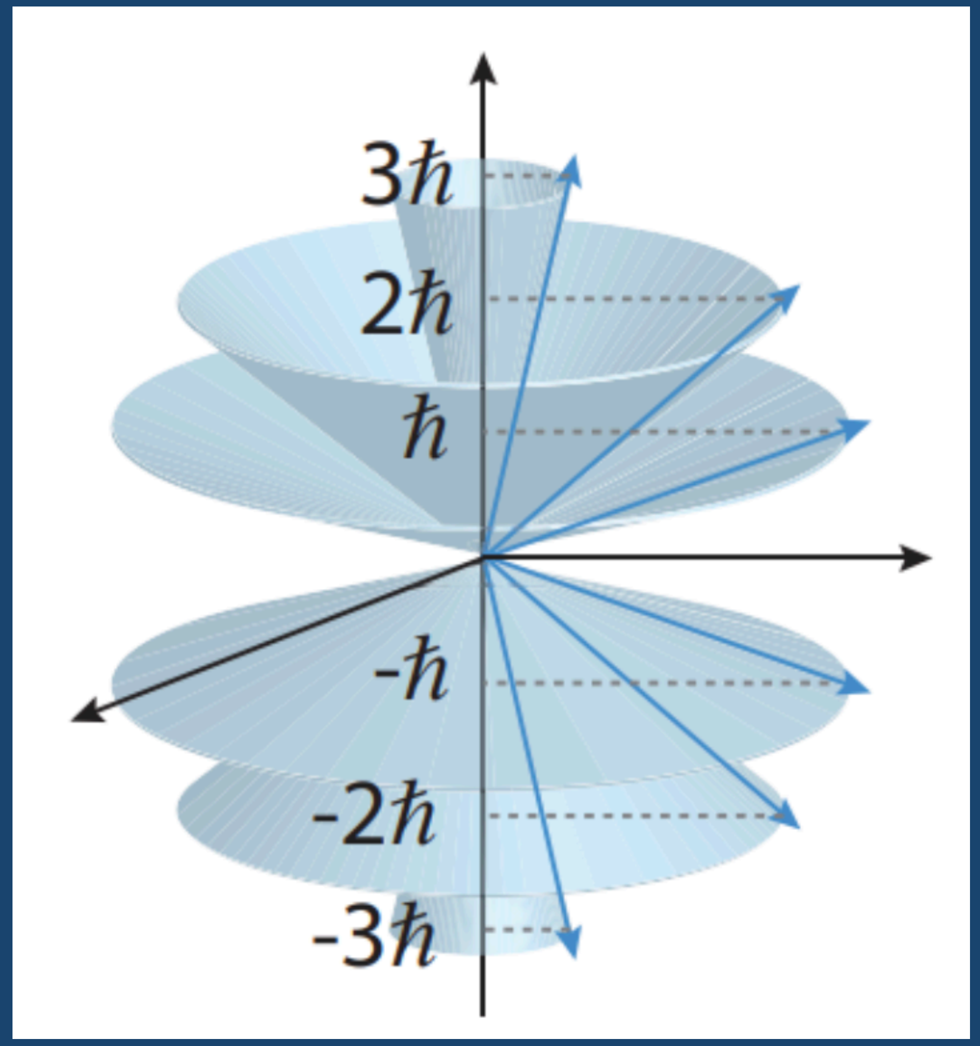

Spin Systems

$SU(2)$ irreps

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Applications in simulating physical systems

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Theorem: An N-dimensional irrep of $SU(2)$ can be implemented with quantum circuits of size $\mathrm{polylog}(N)$.

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Theorem: An N-dimensional irrep of $SU(2)$ can be implemented with quantum circuits of size $\text{polylog}(N)$.

Theorem: An N-dimensional **totally symmetric** irrep of $SU(n)$ can be implemented with quantum circuits of size $O(n^2 \, \text{polylog}(N))$.

Jordan-Schwinger map

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Realize **SU(2)** generators as a quadratic in bosonic operators:

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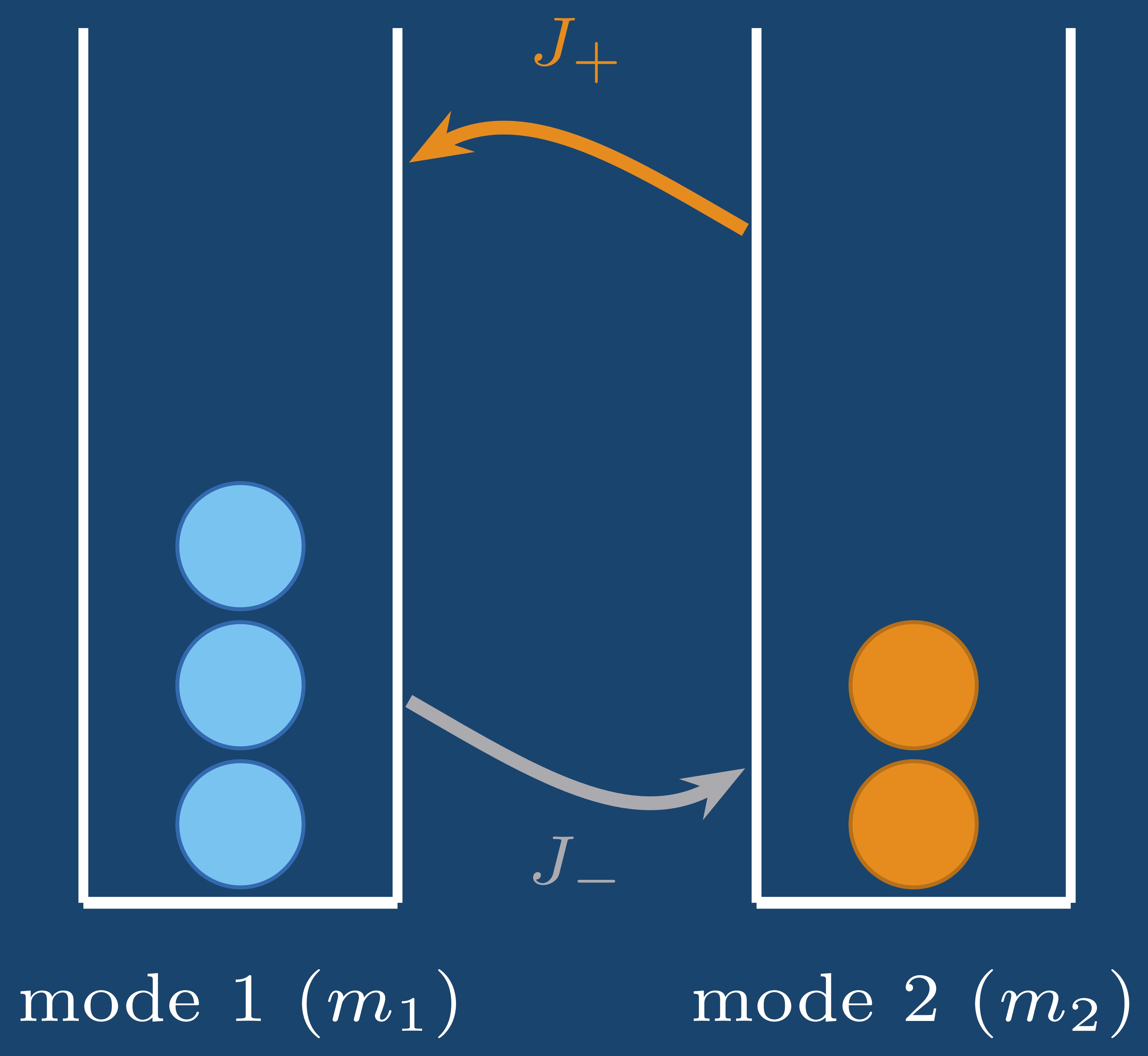

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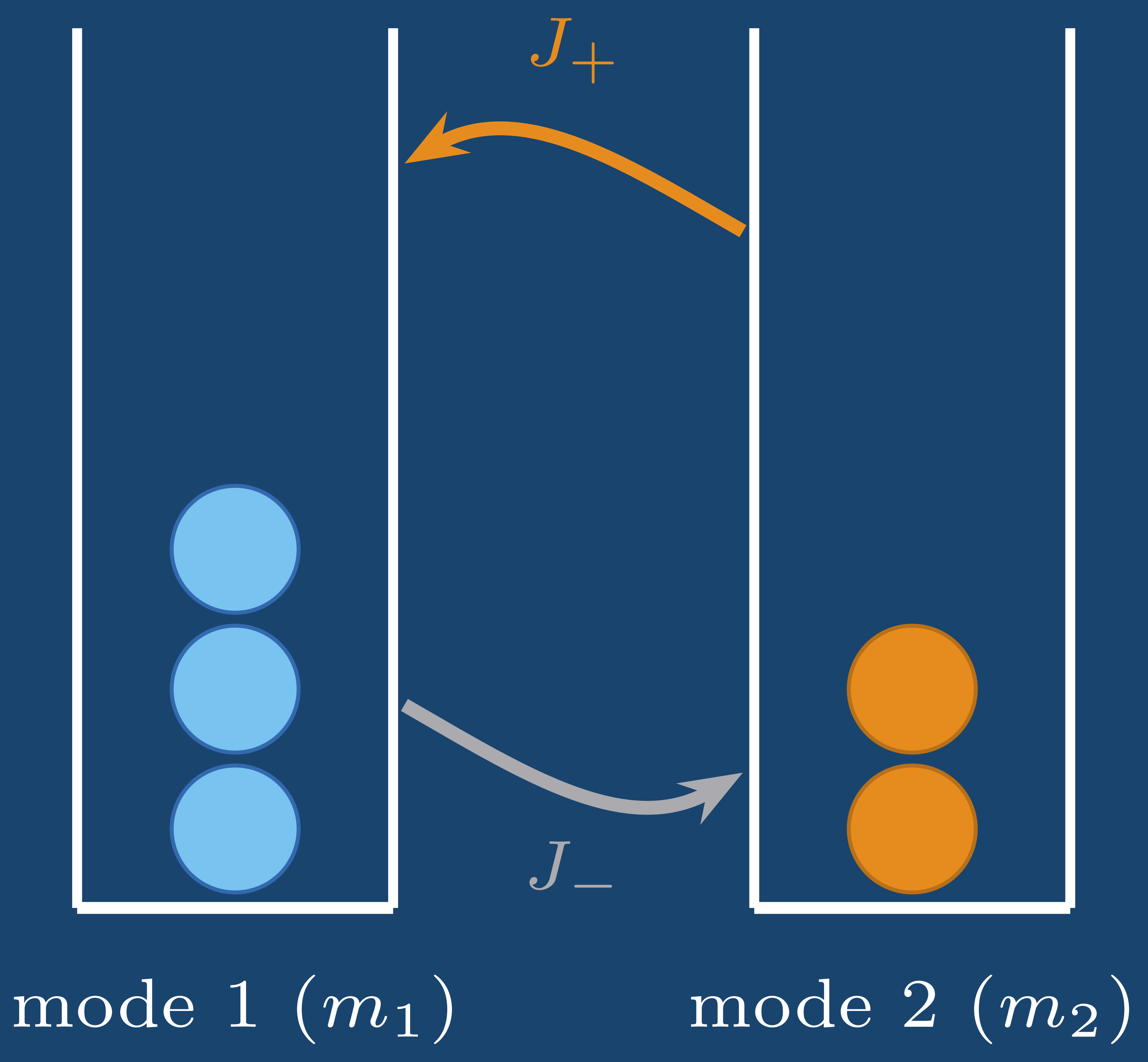
Fix total boson number $M = N - 1$: spin-$M/2$ representation, dimension $N = M + 1$.



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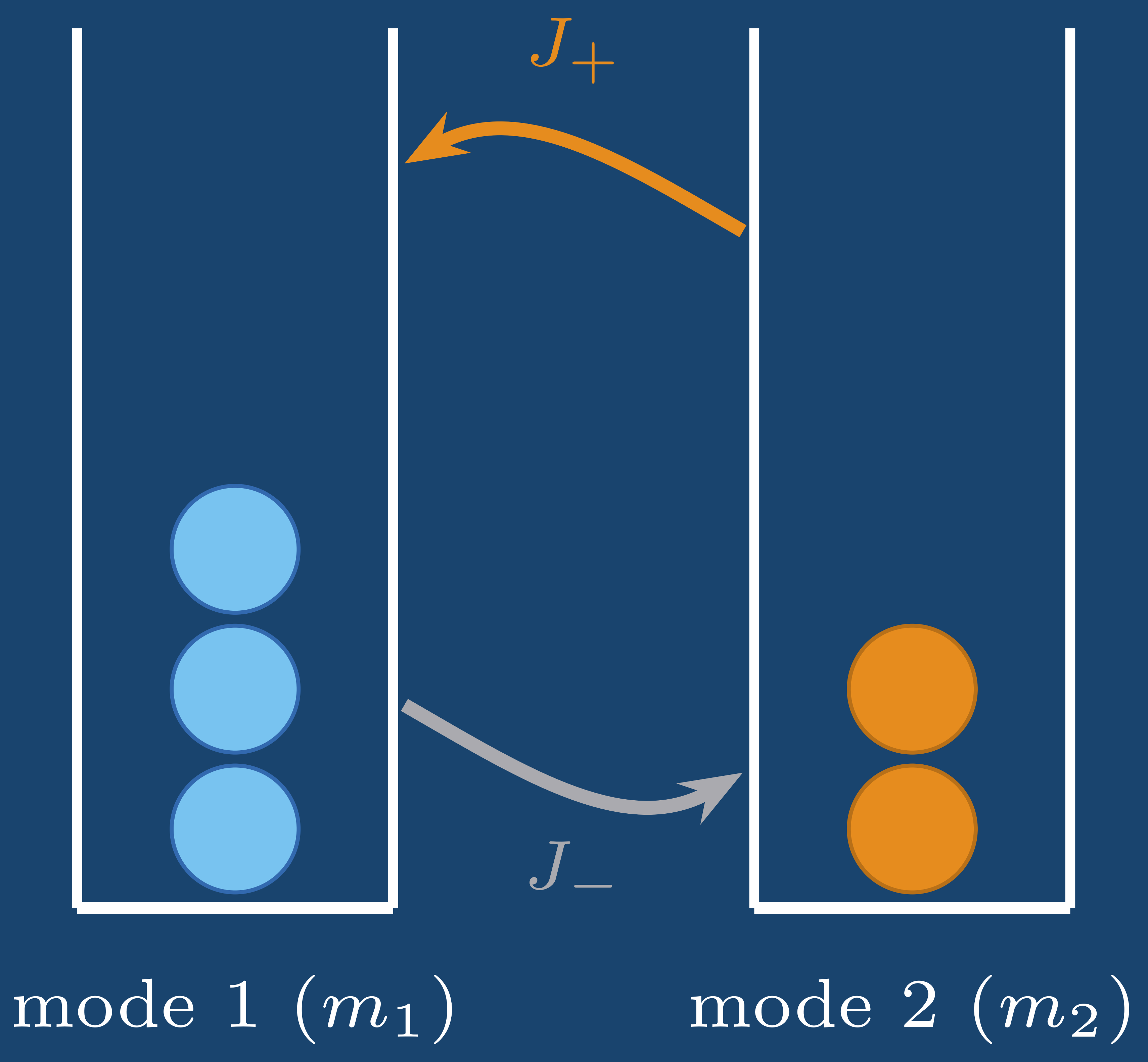
$\exp(-i\theta J_x)$ is a **beam-splitter** between two oscillator modes.



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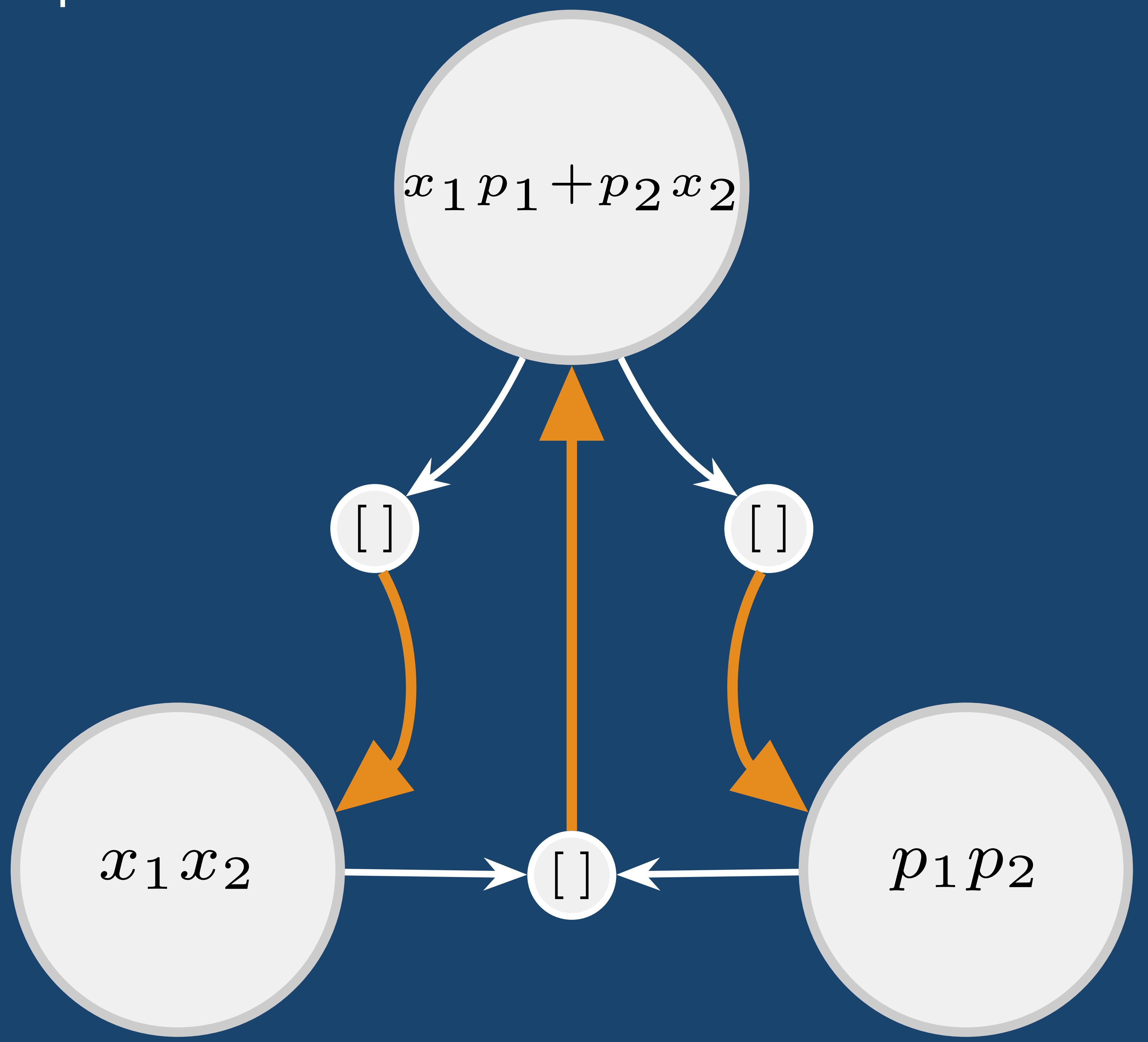

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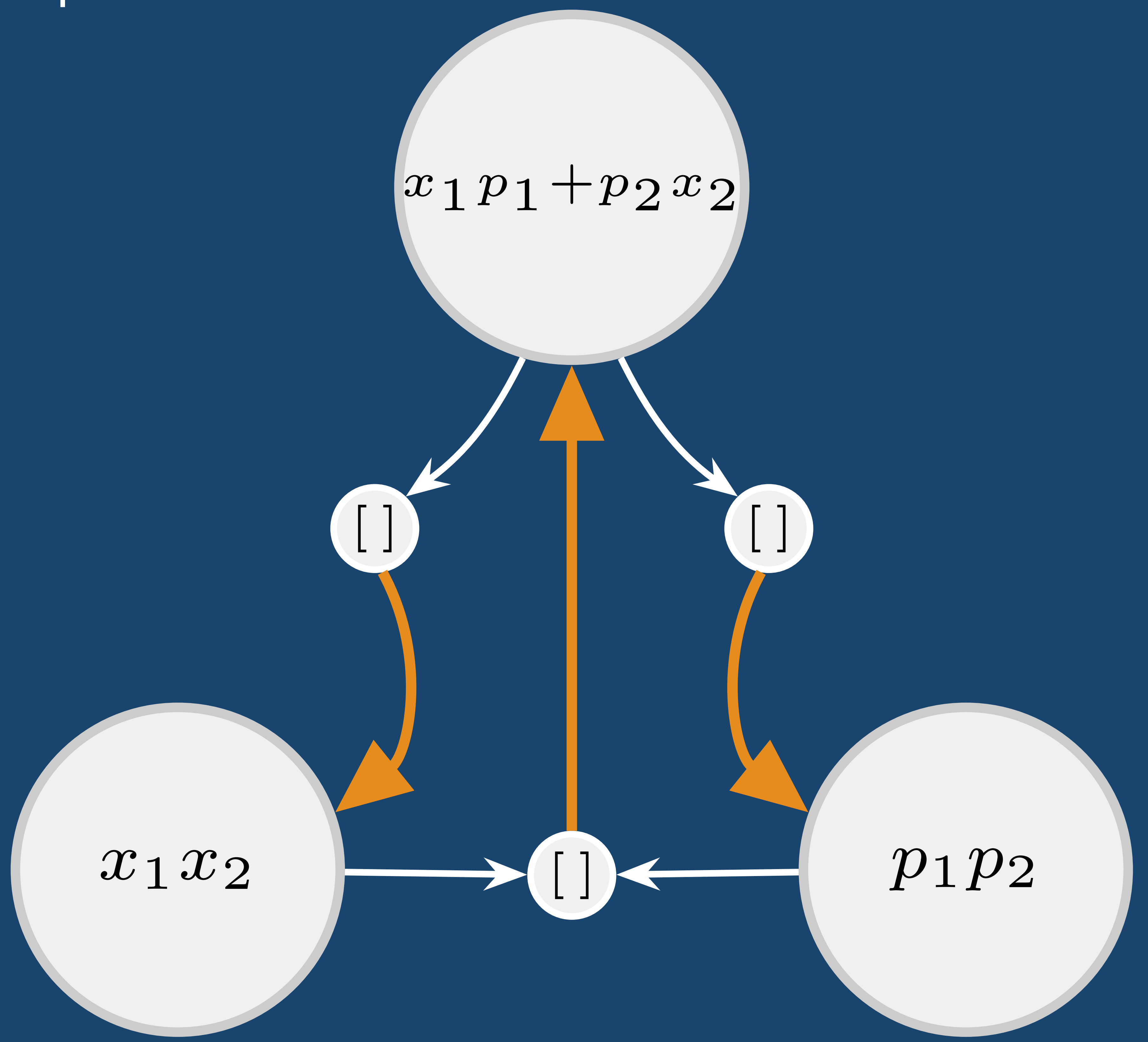

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So we get this operation for **free**!

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Cartan-Weyl basis

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$$U := \exp\left(\mathrm{i}\left(\sum_{i=1}^{n-1} \varsigma_i H_i + \sum_{1\leq j<k\leq n} \vartheta_{j,k} S_{j,k} + \varphi_{j,k} A_{j,k}\right)\right)$$

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$V \in SU(n)$ $\longrightarrow$ $\zeta_i,\ \vartheta_{j,k},\ \varphi_{j,k}$

Givens

Rotation

Approach

Approach

Totally symmetric irreps of $SU(n)$

Approach

Totally symmetric irreps of $SU(n)$ $\longleftrightarrow$ n harmonic oscillators

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$$\exp(\mathrm{i}\overline{O}_a) = \sum_{k=0}^{\infty} \frac{\mathrm{i}^k \overline{O}_a^k}{k!} = \sum_{k=0}^{K} \frac{\mathrm{i}^k (\theta_a \overline{O}_a)^k}{k!} + \sum_{k=K+1}^{\infty} \frac{\mathrm{i}^k (\theta_a \overline{O}_a)^k}{k!}$$

$$\approx \sum_{k=0}^{K} \frac{\mathrm{i}(\theta_a \hat{O}_a)^k}{k!}$$

Small "leakage"

PART 02

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Expanders

Quantum Expanders

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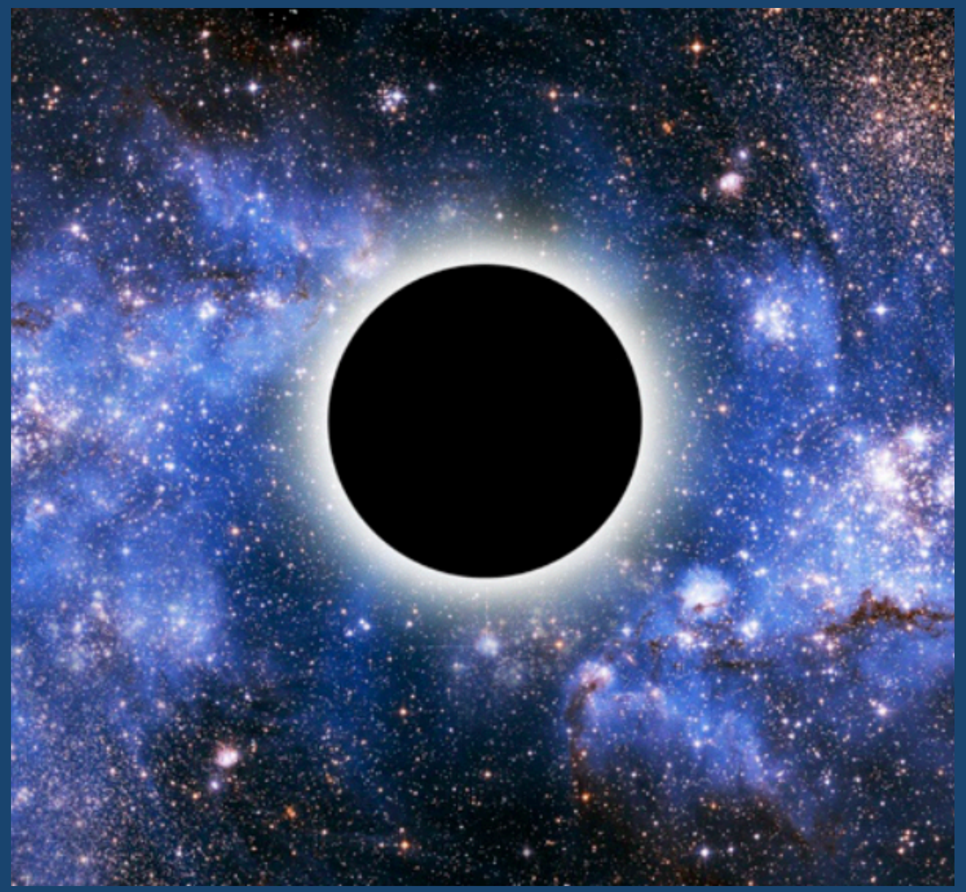

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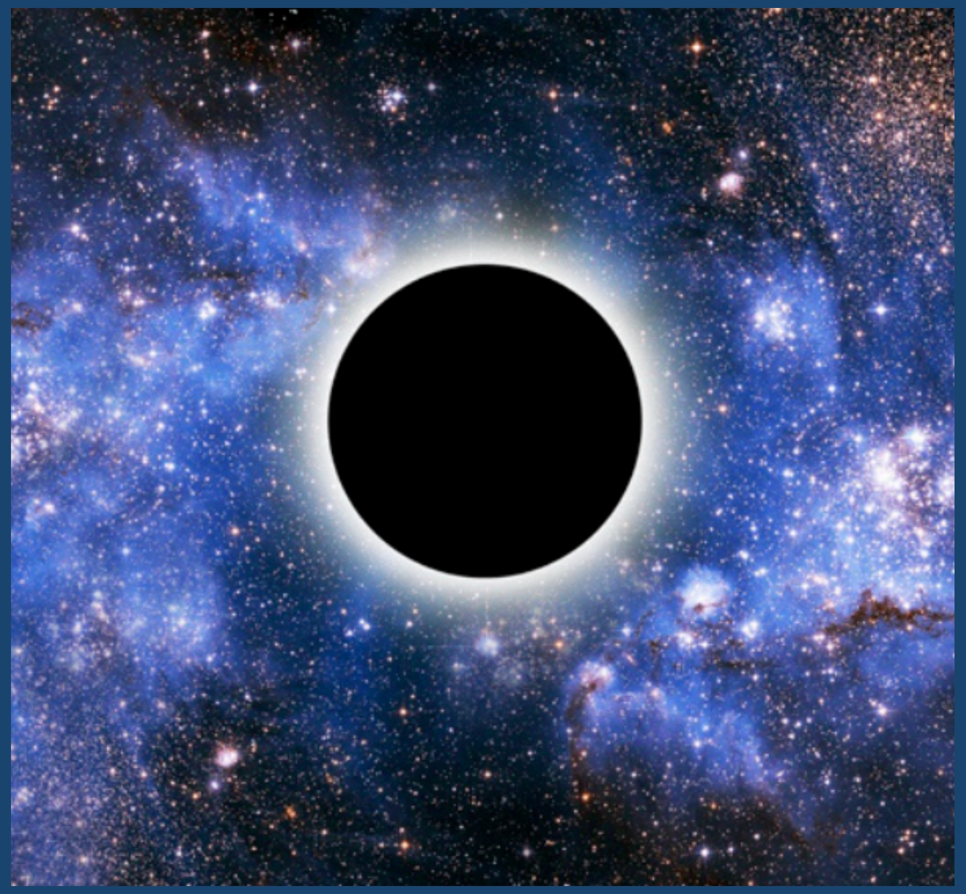

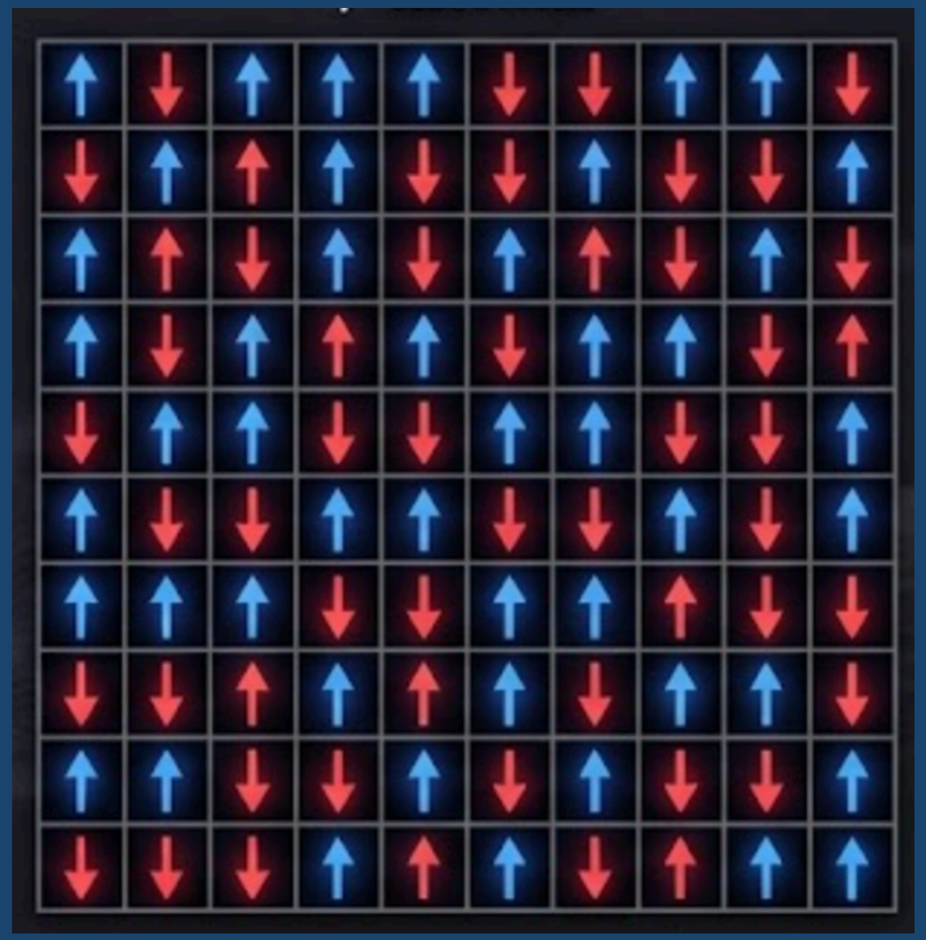

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Irreps of $SU(2)$ → Ramanujan quantum expanders

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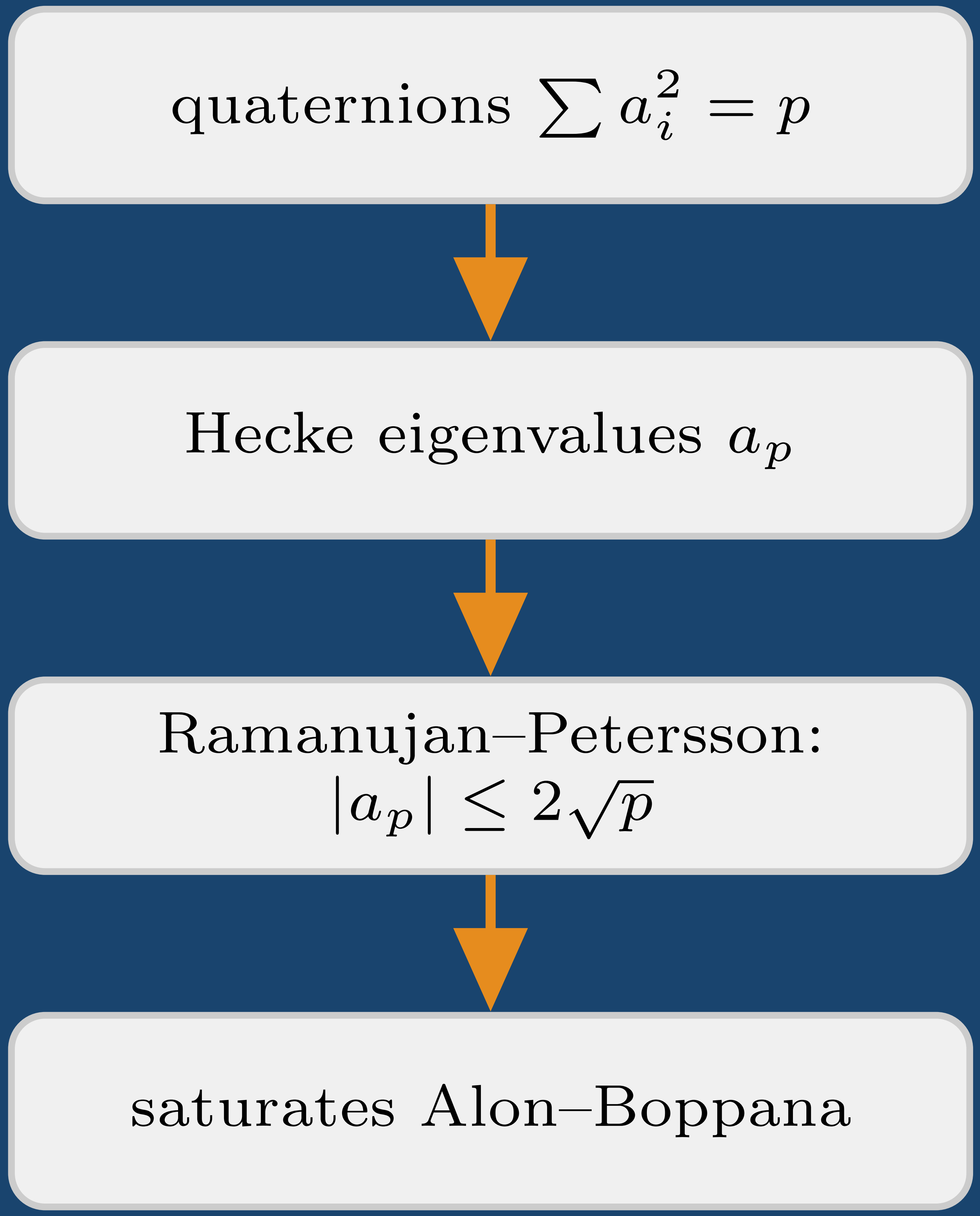

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The reduction chain:

quaternions $\sum a_i^2 = p$

↓

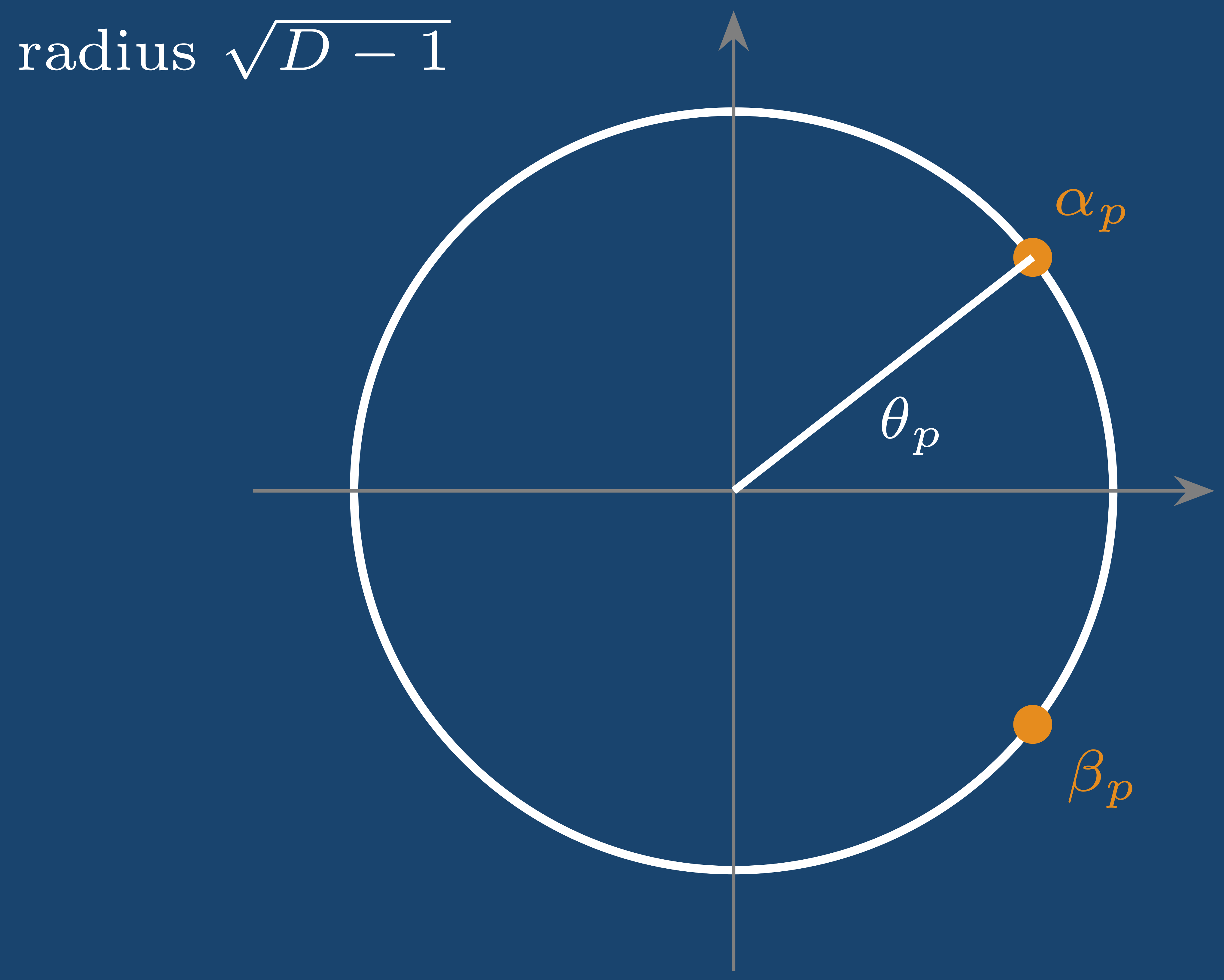
Hecke eigenvalues a_p

↓

Ramanujan–Petersson: $|a_p| \leq 2\sqrt{p}$

↓

saturates Alon–Boppana



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Thanks for your attention!

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