On Pigeonhole Principles and Ramsey in TFNP

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joint work with

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UT Austin

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McGill

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UT Austin

Total Function NP

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NP search problems which **always** have a sol^n

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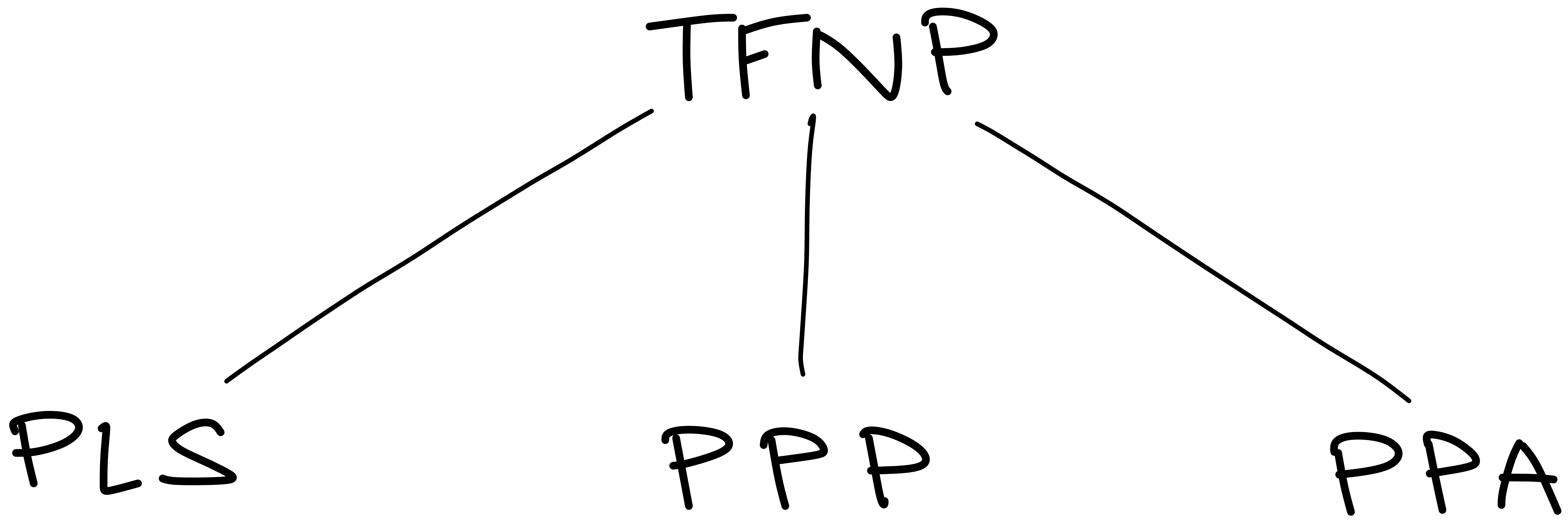

Total Function NP

NP search problems which **always** have a sol^{n}

TFNP

PLS

every DAG has a **sink**

PPP

every compressing map has a **collision**

PPA

every line has a **sink**

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How constructive are these combinatorial principles?

$N = 2^n$

A "Rogue" problem

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K-RAMSEY

Input $[N] \times [N] \to \{0,1\}$

Solutions

↳ Directed edges

↳ Self loops

↳ K-clique or independent set

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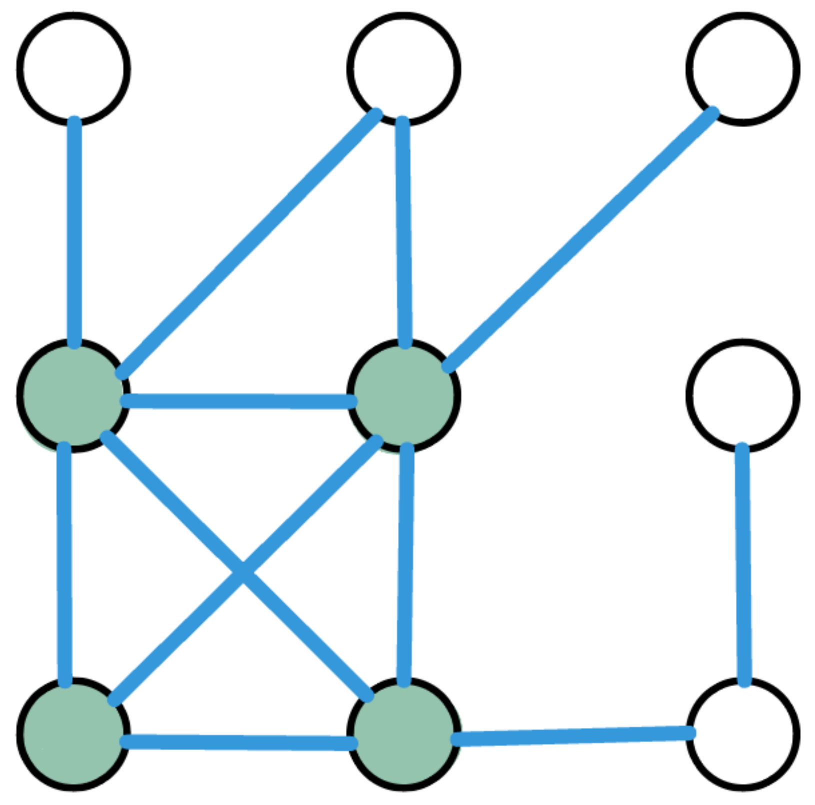

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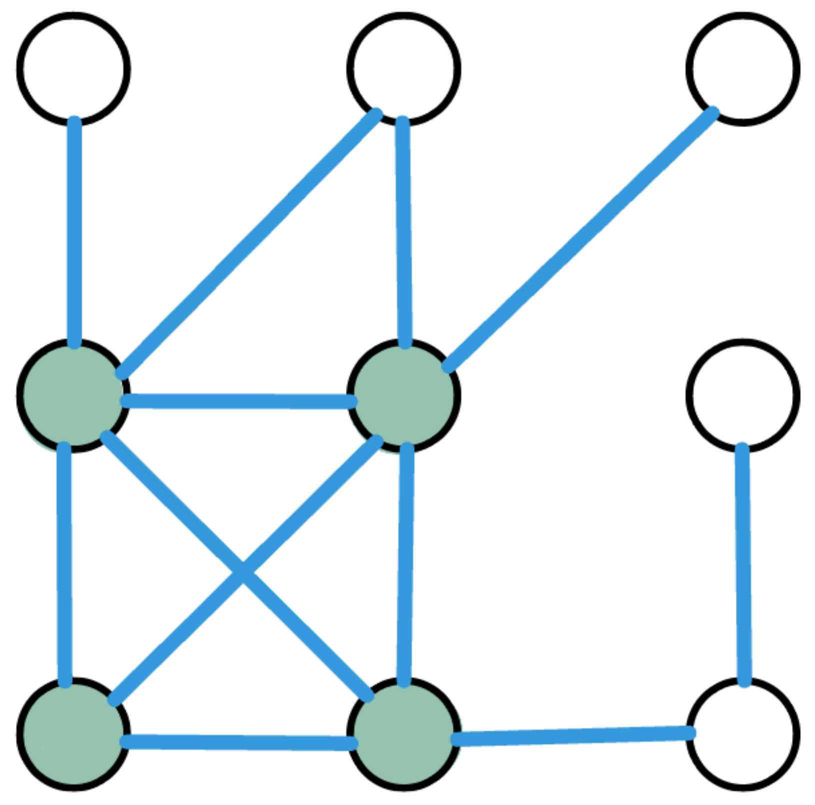

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Ramsey's Theorem

K-RAMSEY is total

for $K = \frac{n}{2}$

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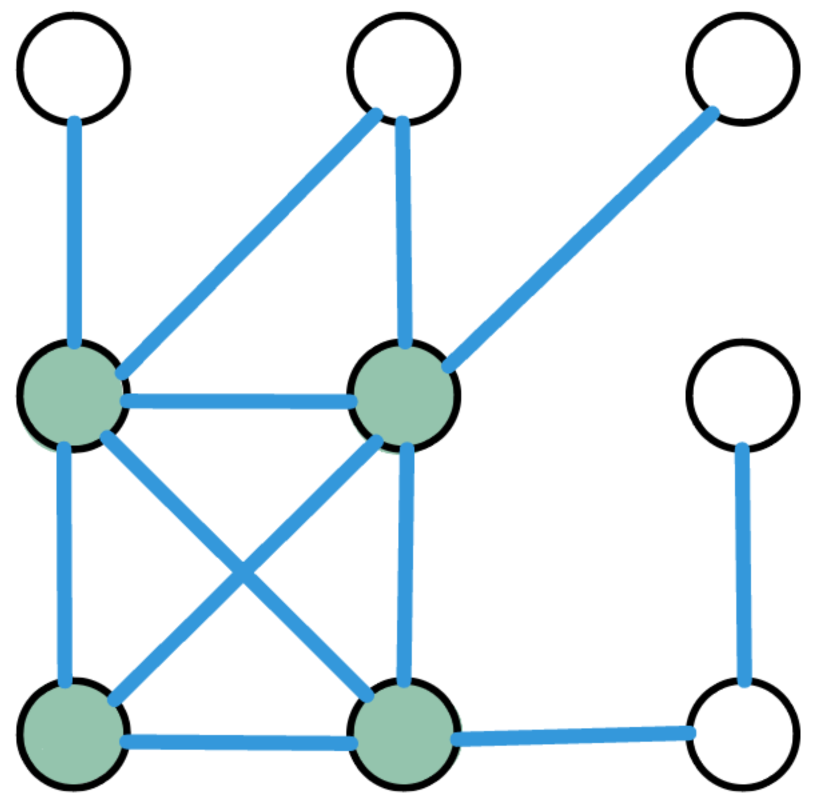

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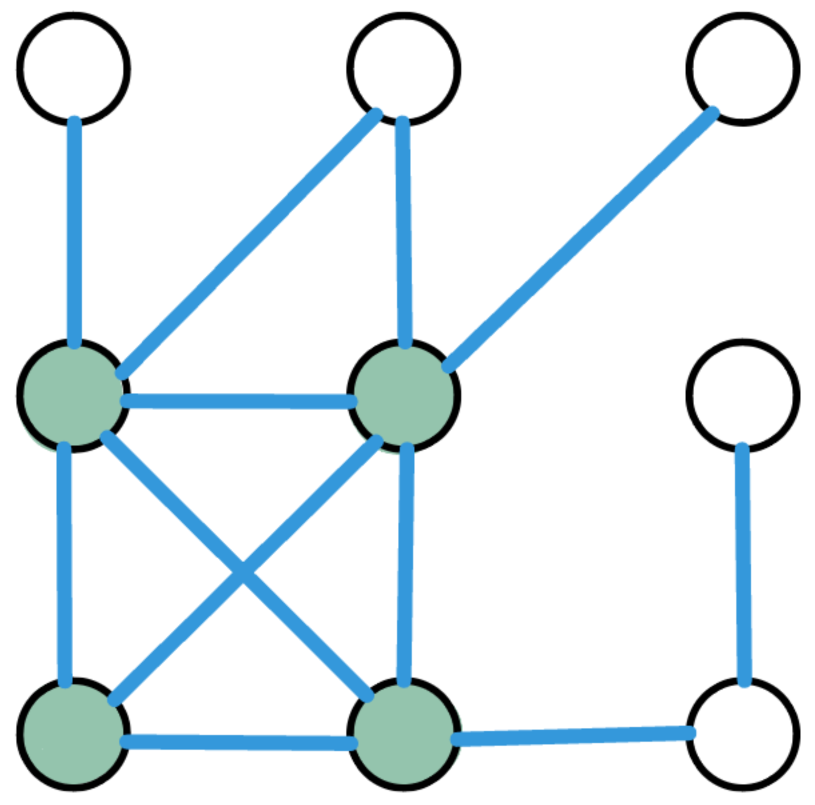

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Conjecture [GP'17]

$\frac{n}{2}$-RAMSEY $\in$ PPP

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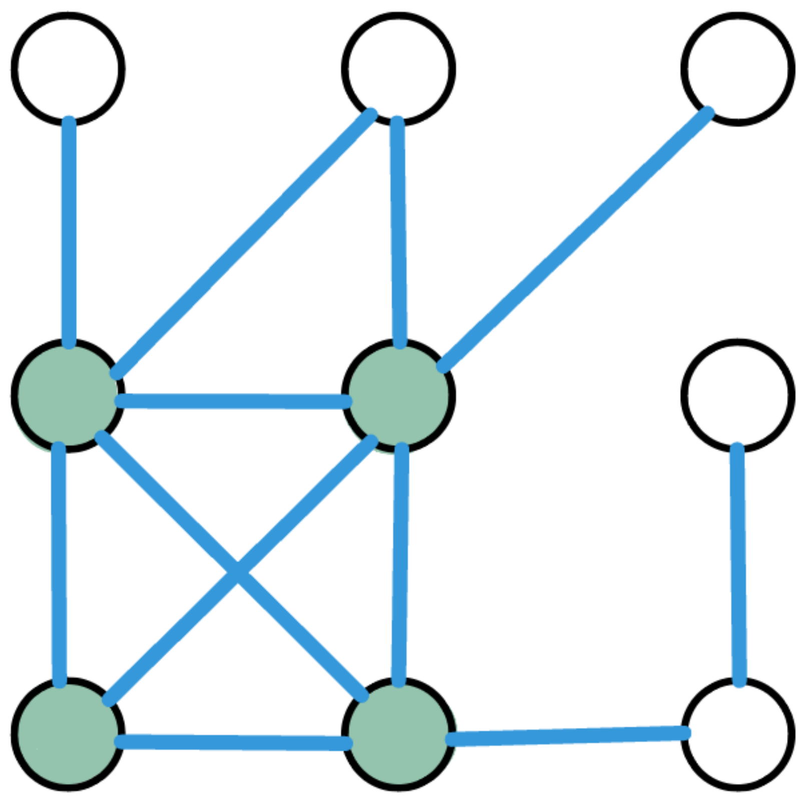

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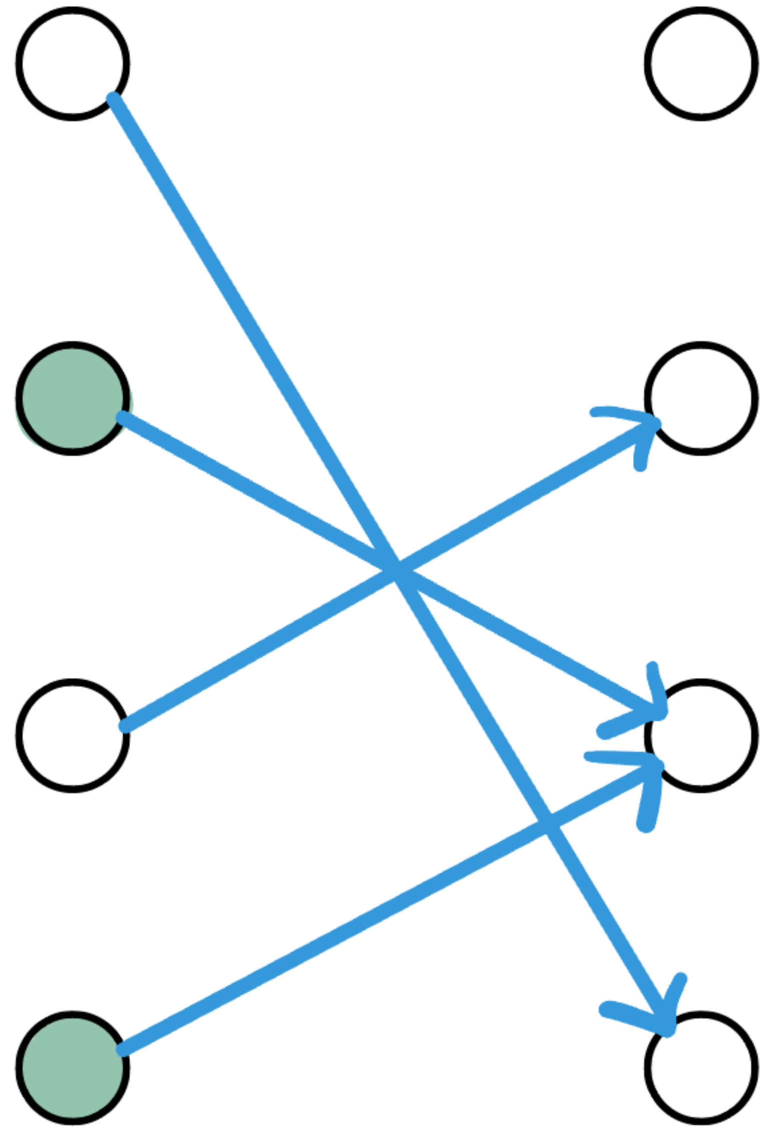

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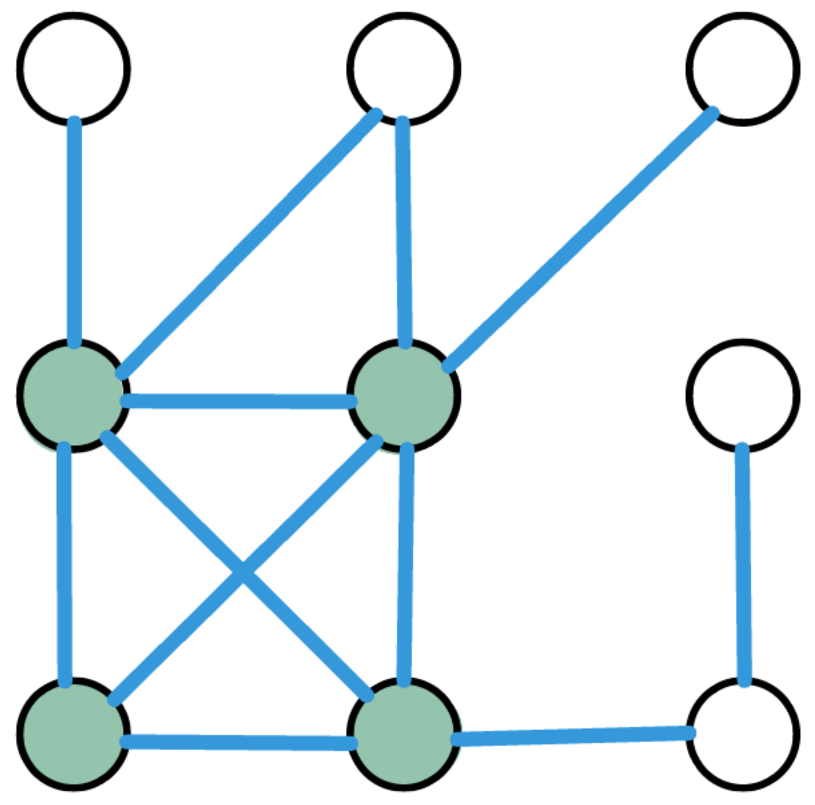

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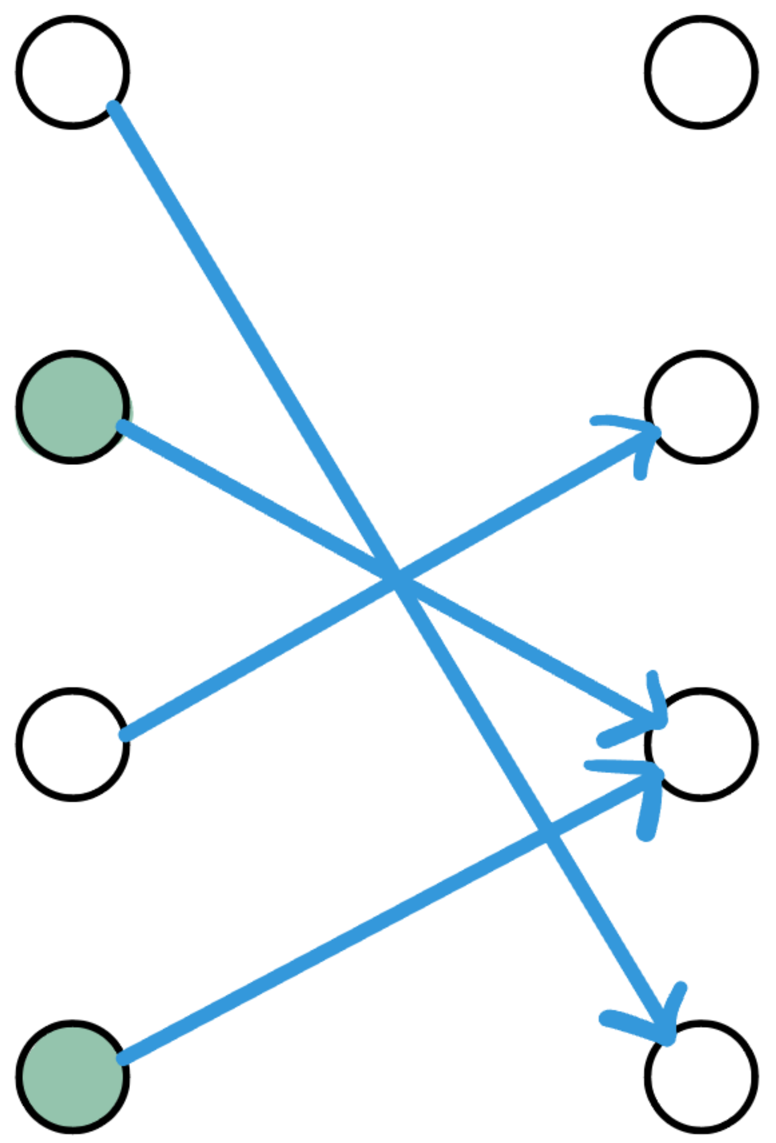

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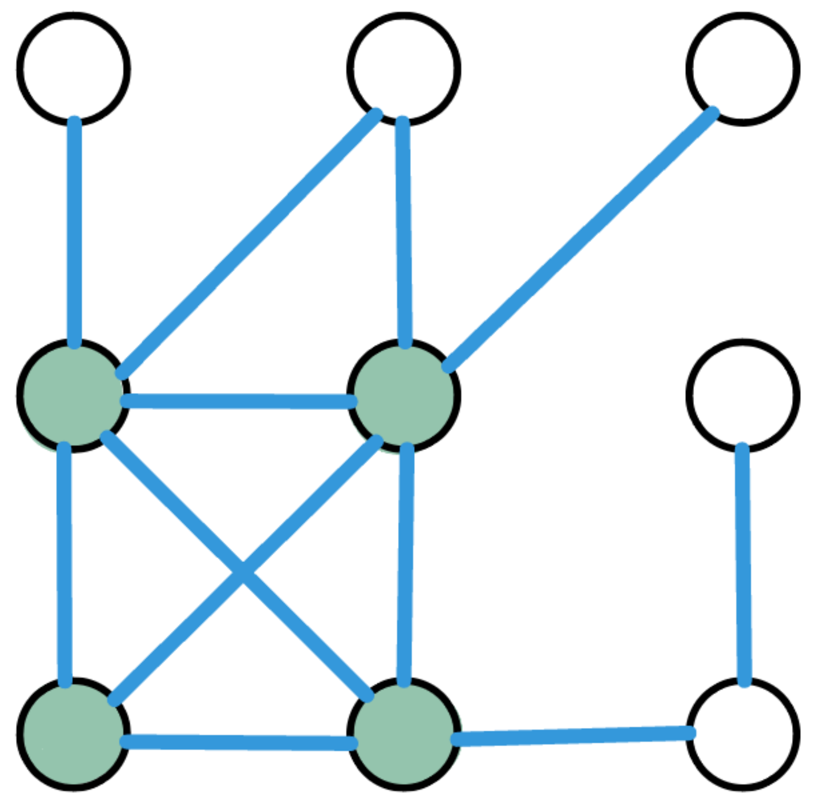

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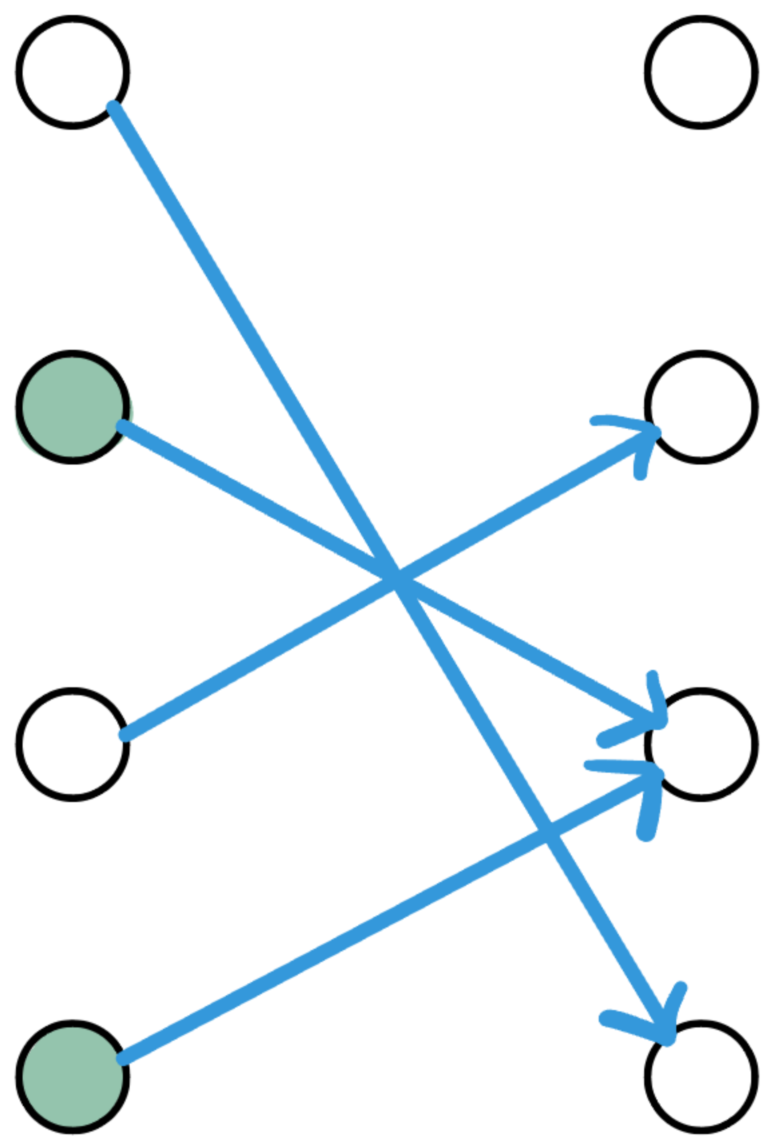
FALSE (BLACK-BOX)

OUR WORK



Generalized Pigeonhole Principle

t-PIGEON$_N$

Input $[M] \to [N]$ $\quad (M = (t-1)N)$

Solutions $\to$ $t-1$ pigeons mapped to 0

$\to$ t collision

$N = 2^n$

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Input $[M] \to [N]$ $(M = (t-1)N)$

Solutions → $t-1$ pigeons mapped to 0

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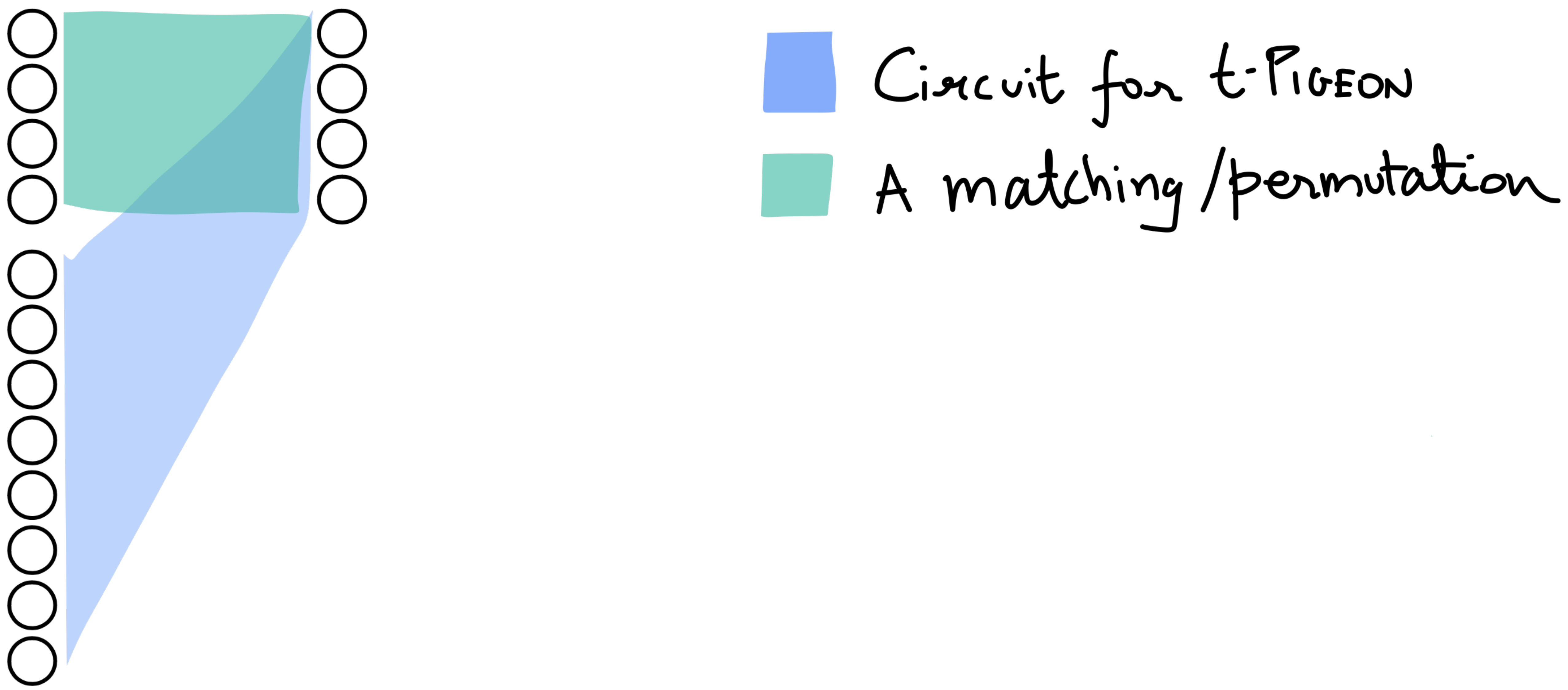
Lemma For any $t(n)$, t-PIGEON$_N \le (t+1)$-PIGEON$_N$

Proof



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Polynomial Averaging Principle (PAP)

Everything reducible to n-PIGEON$_N$

Lemma For any $t(n)$, t-PIGEON$_N \leq (t+1)$-PIGEON$_N$

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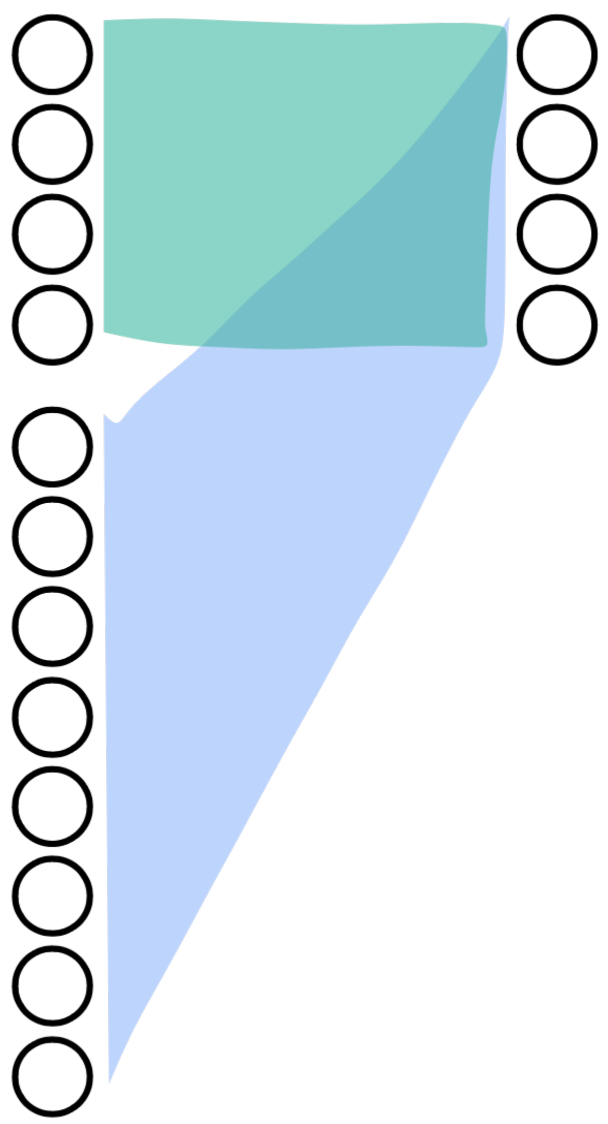

Circuit for t-PIGEON

A matching/permutation

$N = 2^n$

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equivalent to poly(n)-PIGEON$_N$

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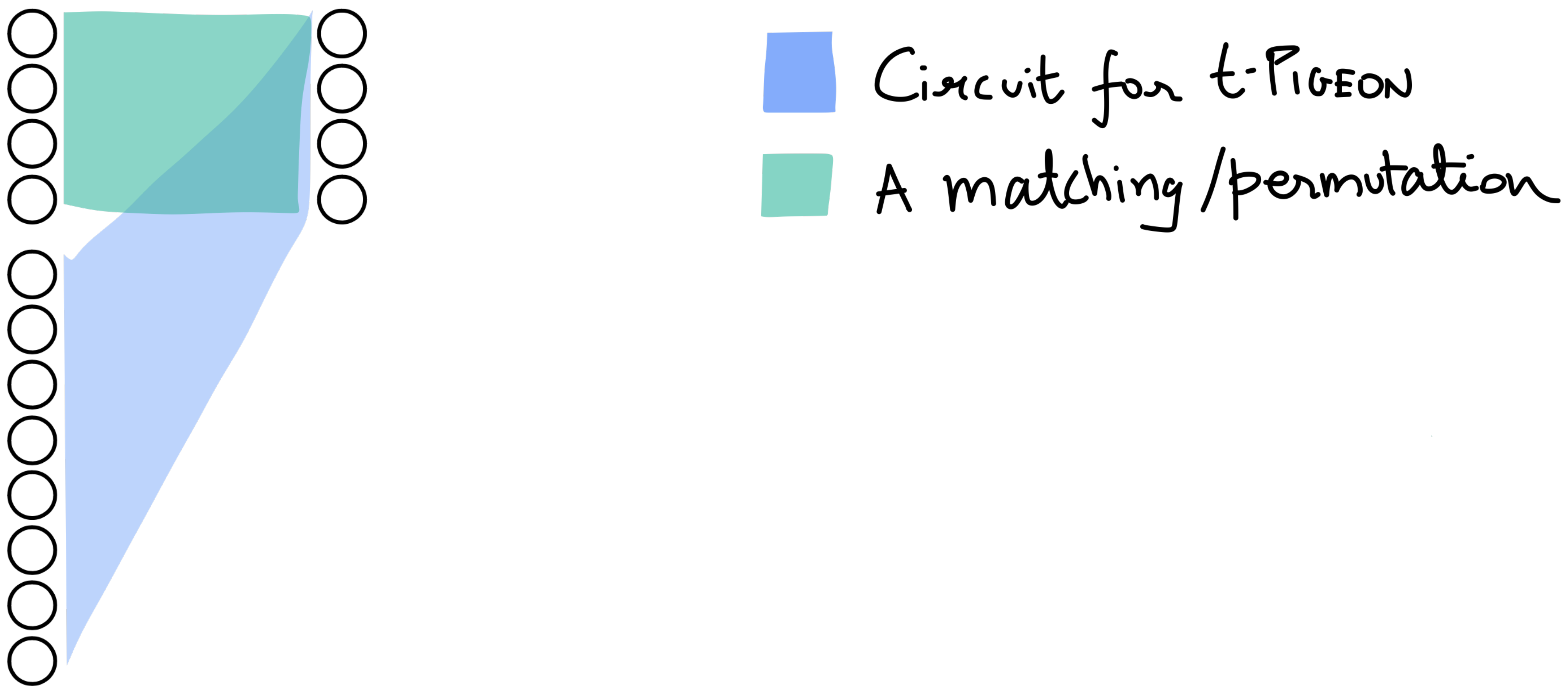

$N = 2^n$

~~Generalized Pigeonhole Principle~~

Pecking Order

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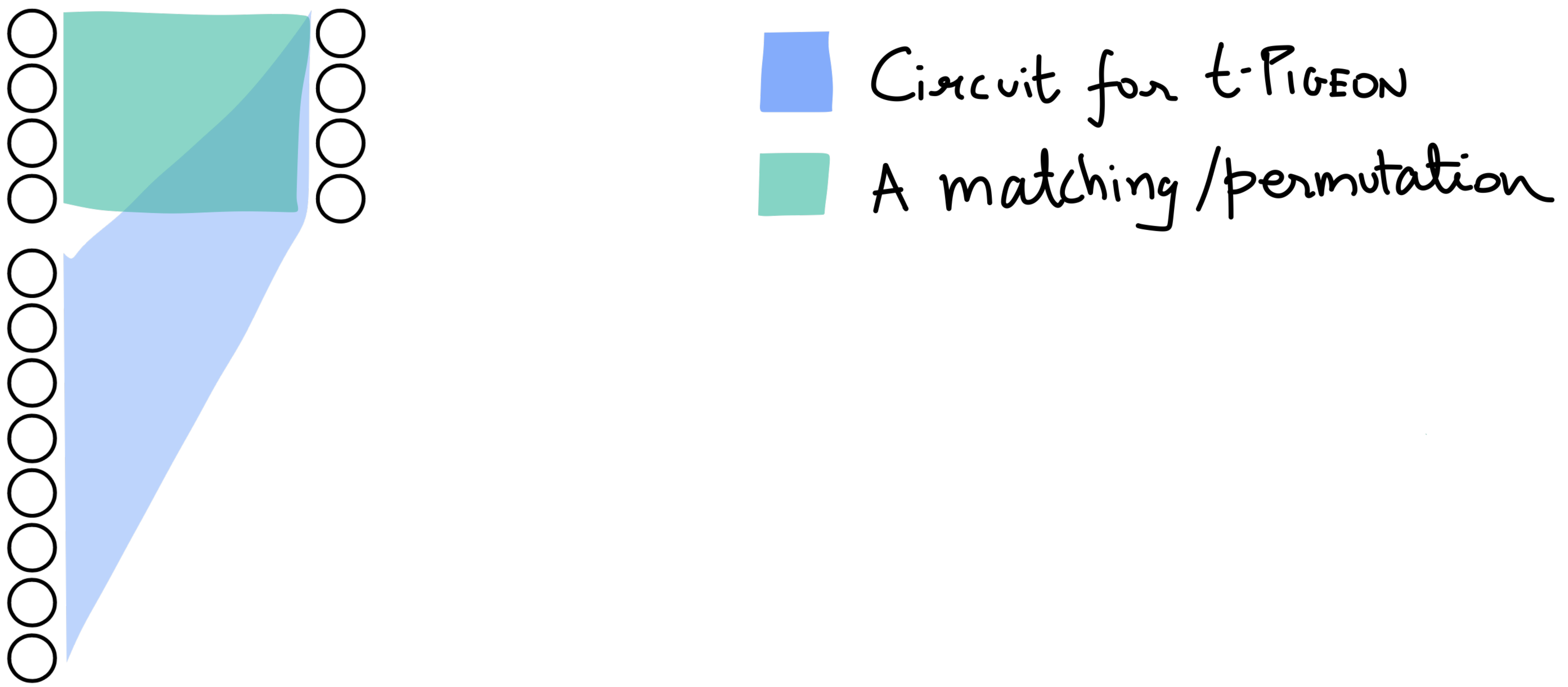

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A Reduction

$N = 2^n$
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A Reduction

Theorem $3\text{-PIGEON}_N^M \leq \text{RAMSEY}_M$ when $M \geq N^{12}$

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Proof $h : [M] \to [N]$

G_0 on N vertices with no C/IS of size $2n$

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$$(u,v) \in E \begin{cases} h(u) = h(v) \\ (h(u), h(v)) \in E_0 \end{cases}$$

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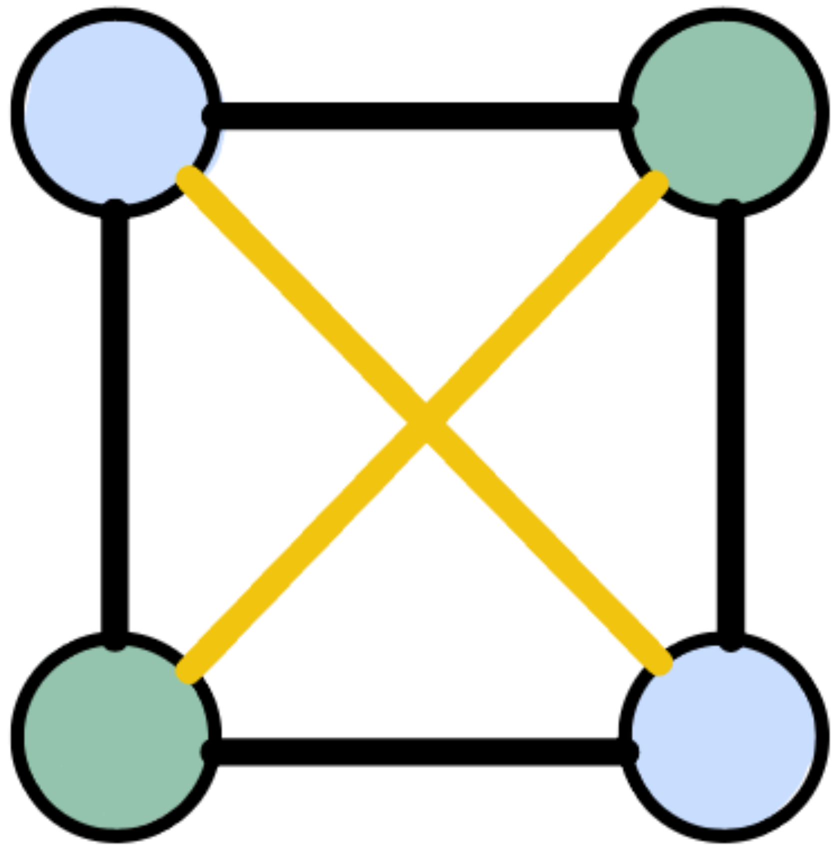

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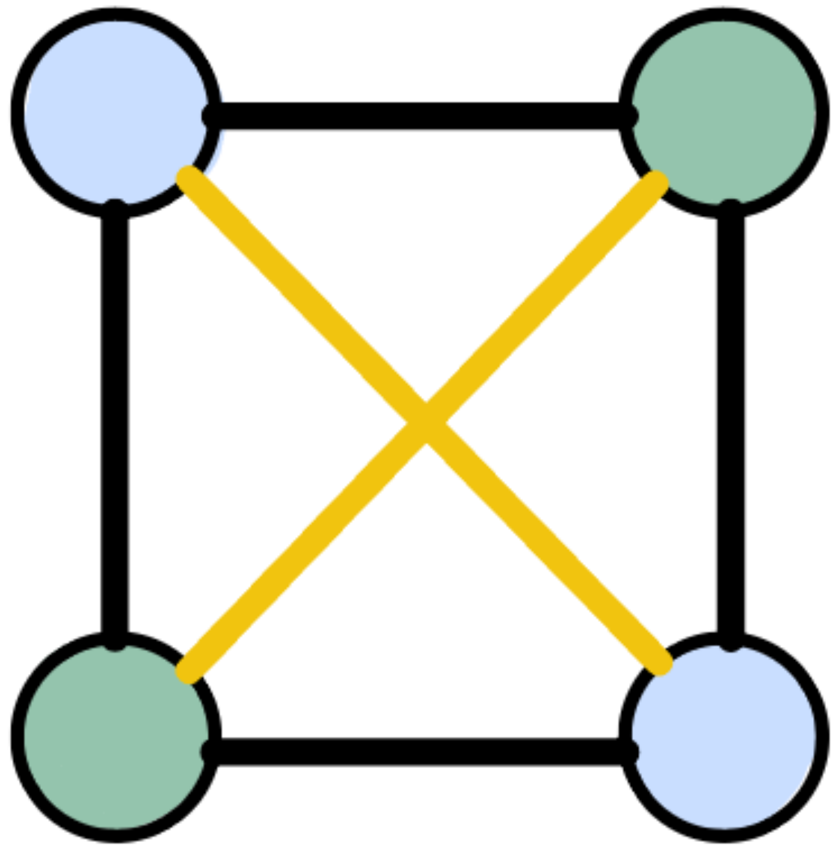
Proof

$$(u,v) \in E \begin{cases} h(u) = h(v) \\ (h(u), h(v)) \in E_0 \end{cases}$$

$6n$-clique in G has at most

$2n-1$ distinct vertices $h(u)$



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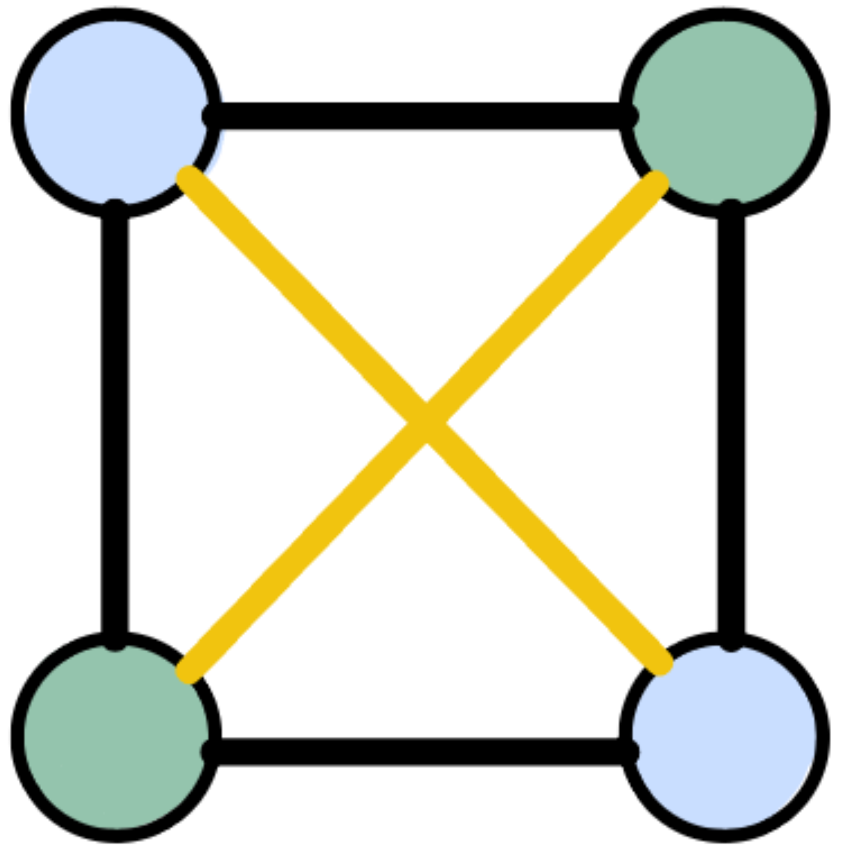
$\therefore$ must contain a $t = \frac{6n}{(2n-1)}$ collision



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A Separation

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Theorem $3\text{-PIGEON}^{M}_{N} \not\leq_{dt} 2\text{-PIGEON}_{N}$

whenever
$m = \text{poly}(n)$

$N = 2^n$
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A Separation

Theorem $3\text{-PIGEON}_N^M \not\leq_{dt} 2\text{-PIGEON}_N$ whenever $m = \mathrm{poly}(n)$

By the reduction,

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Most of the **work** goes into proving this Theorem.

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Full Theorems

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Full Theorems

Theorem When $2t(2n-1) \le m$, $t\text{-PIGEON}^{M}_{N} \le \text{RAMSEY}_{M}$

Proof is the same as special case.

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Thus we can conclude,

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& Subpolynomial Averaging Principle (SAP)

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The Full Picture

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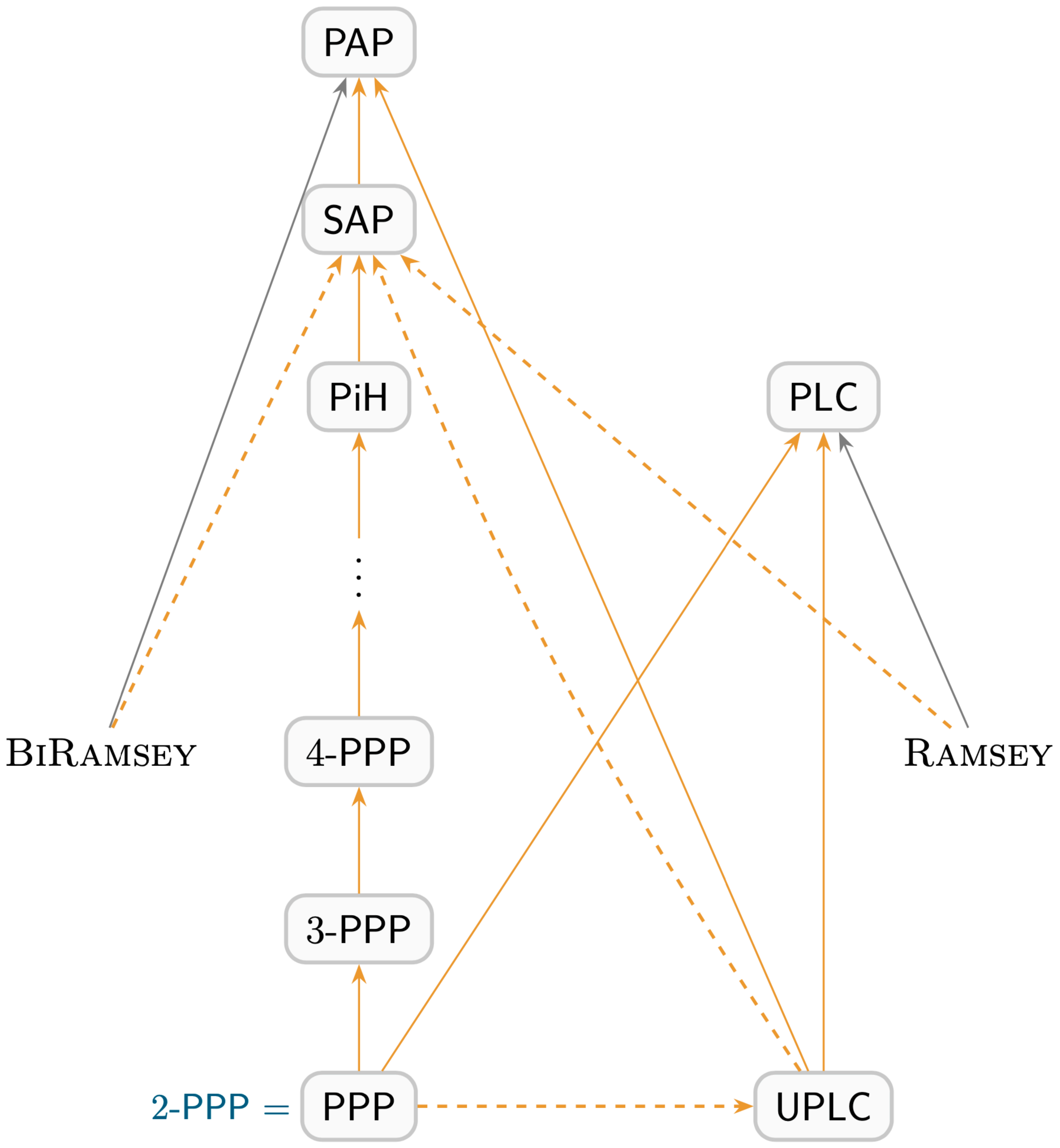

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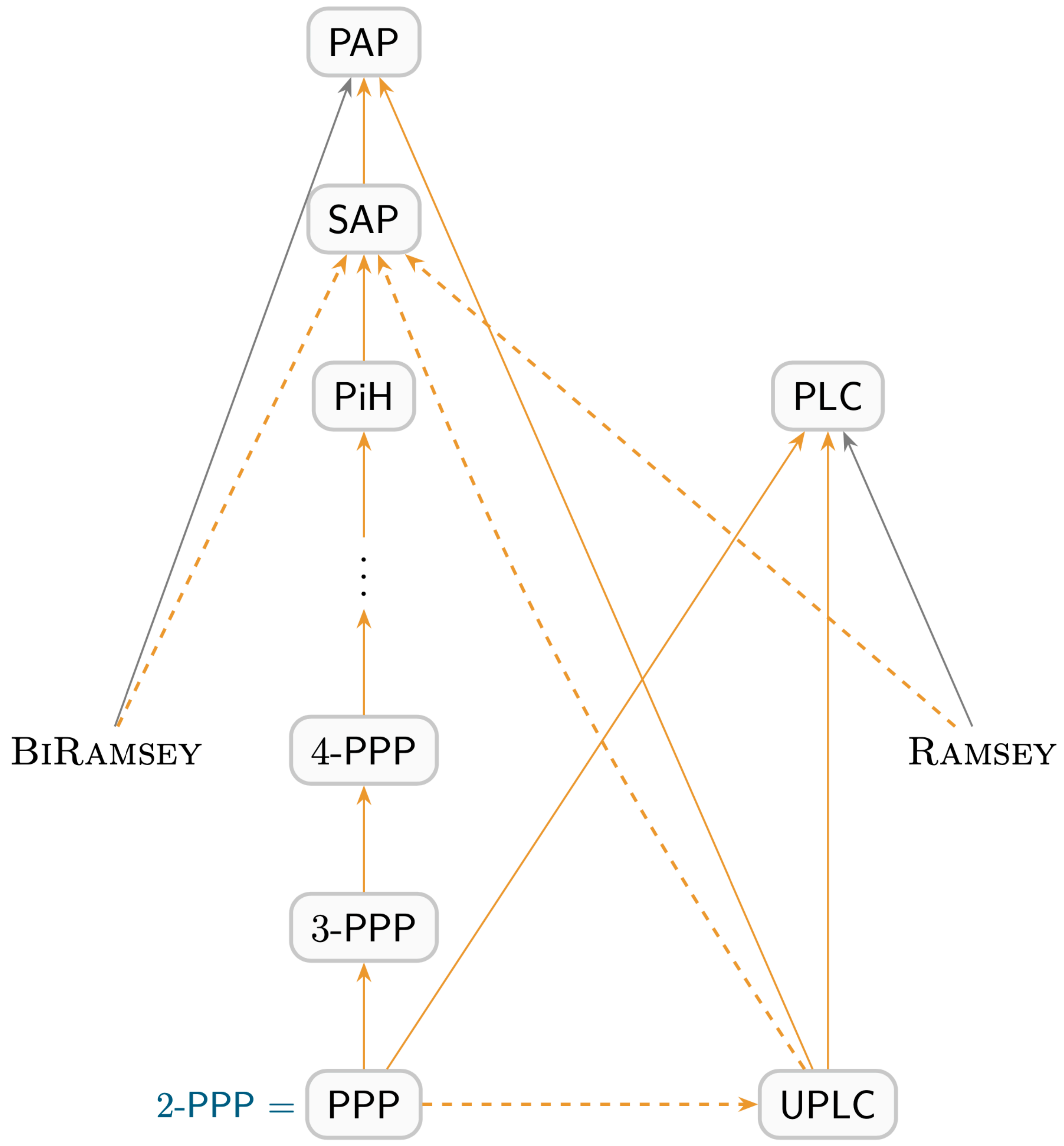
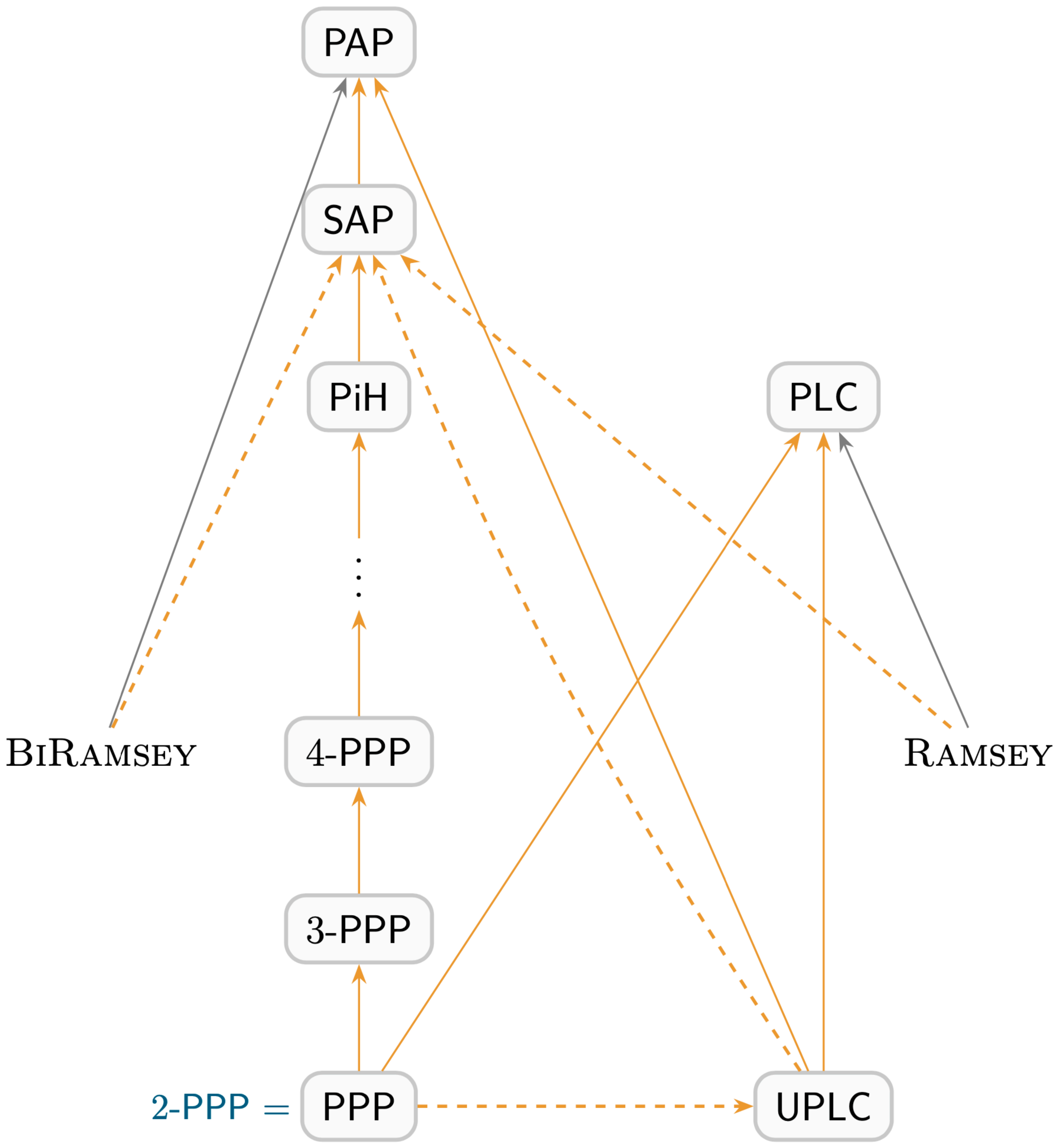


PiH Pigeon Hierarchy

$$= \bigcup_{t=2}^{\infty} t\text{-PPP}$$

The Full Picture



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$A \dashrightarrow B$

$A \not\leq_{dt} B$

$A \longrightarrow B$

$A \lneq_{dt} B$

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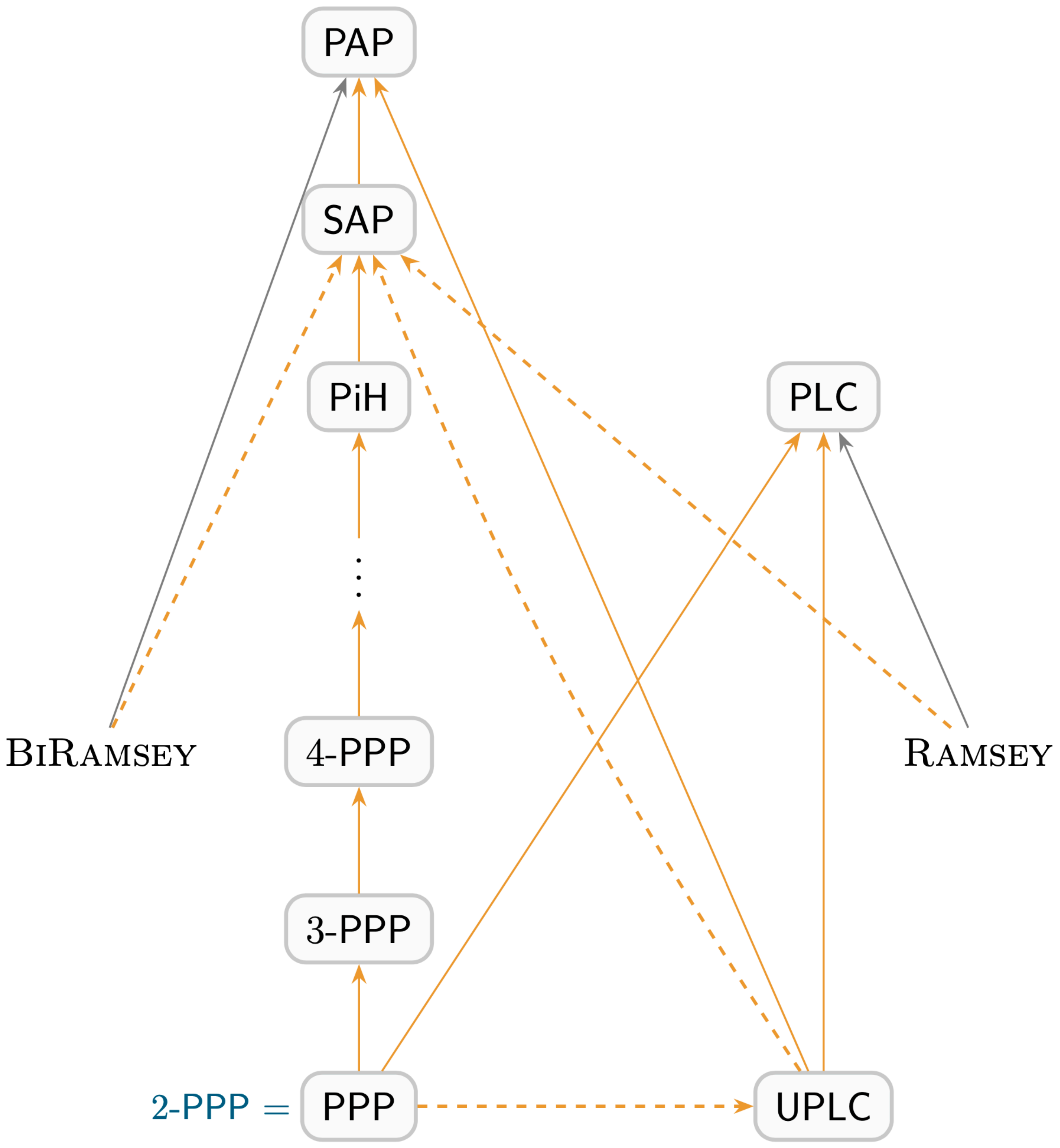

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Everything in orange is

Our Work

Open Problems

$N = 2^n$

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→ RAMSEY $\overset{?}{\leq}$ n - PIGEON$_N$

$N = 2^n$

Open Problems

→ RAMSEY $\overset{?}{\leq}$ n-PIGEON$_N$

aka RAMSEY $\overset{?}{\in}$ PAP

$N = 2^n$

Open Problems

→ $\text{RAMSEY} \overset{?}{\leq} n\text{-PIGEON}_N$

aka $\text{RAMSEY} \overset{?}{\in} \text{PAP}$

→ PAP vs PLC

$N = 2^n$

Thanks for listening!

Au revoir