On Pigeonhole Principles and Ramsey in TFNP

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joint work with

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UT Austin

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McGill

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UT Austin

Total Function NP

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NP search problems which **always** have a sol^n

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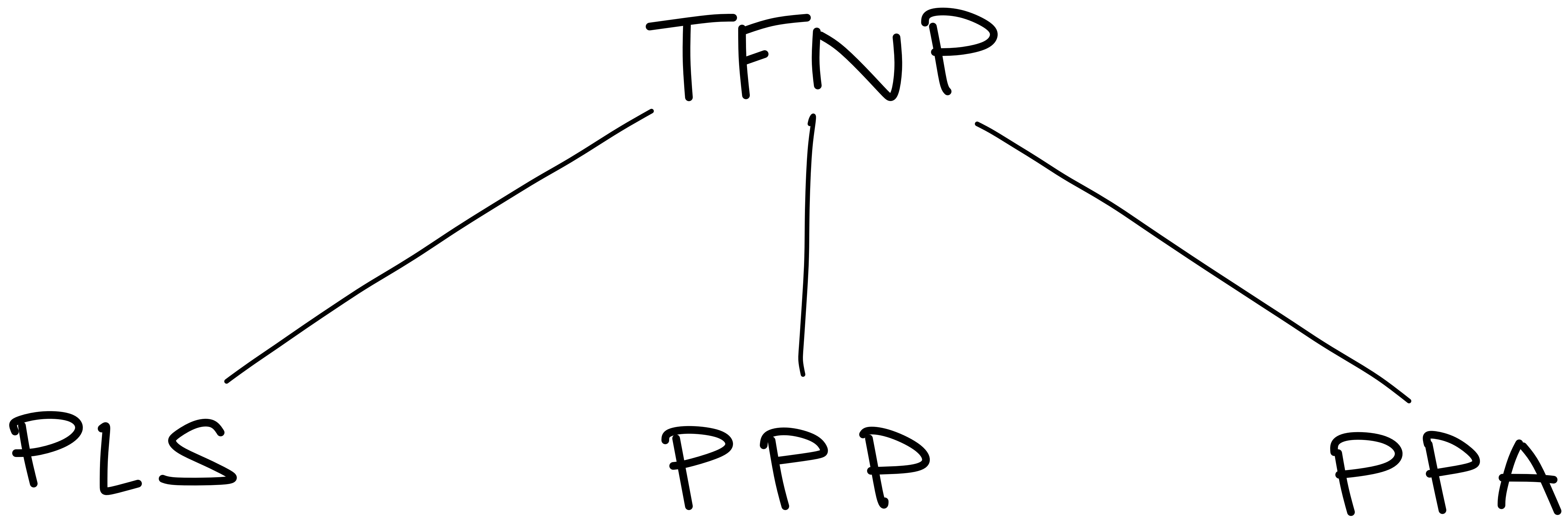

Total Function NP

NP search problems which **always** have a sol^n

- TFNP
  - PLS: every DAG has a **sink**
  - PPP: every compressing map has a **collision**
  - PPA: every line has a **sink**

Total Function NP

NP search problems which **always** have a sol^n

TFNP

PLS

every DAG has a **sink**

PPP

every compressing map has a **collision**

PPA

every line has a **sink**

How constructive are these combinatorial principles?

$N = 2^n$

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A "Rogue" problem

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A "Rogue" problem

K-RAMSEY

Input $[N] \times [N] \to \{0, 1\}$

Solutions

↳ Directed edges

↳ Self loops

↳ K-clique or independent set

$N = 2^n$

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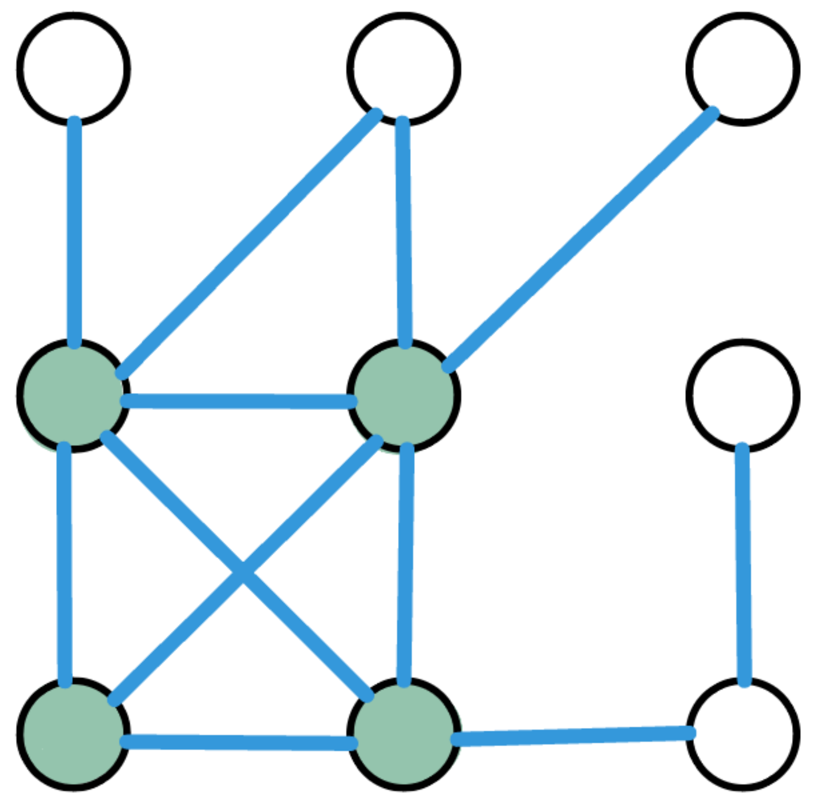

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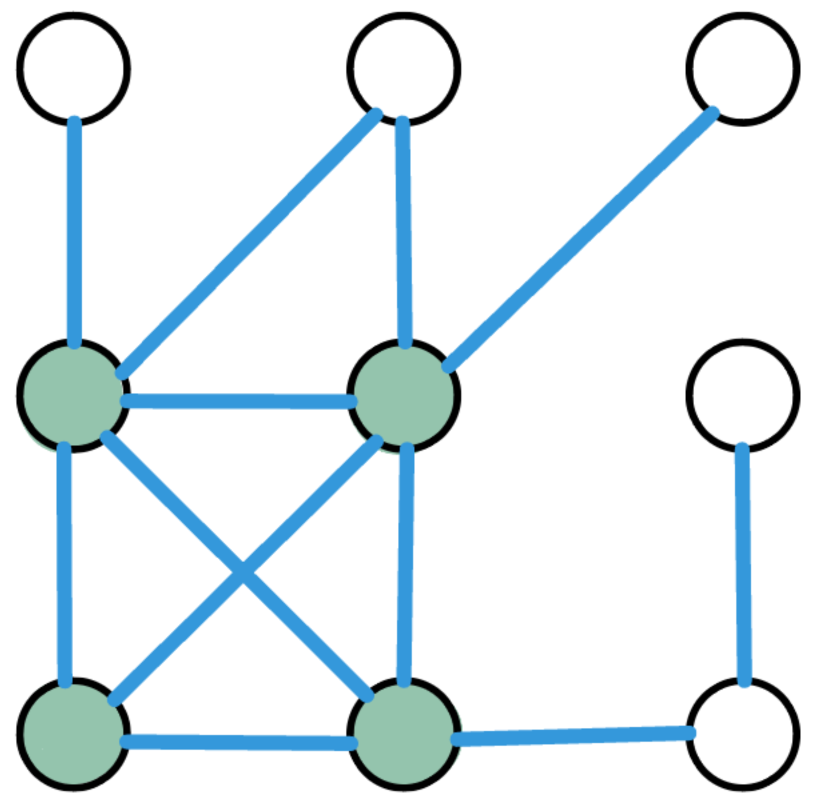

Ramsey's Theorem

K-RAMSEY is total for $K = \frac{n}{2}$

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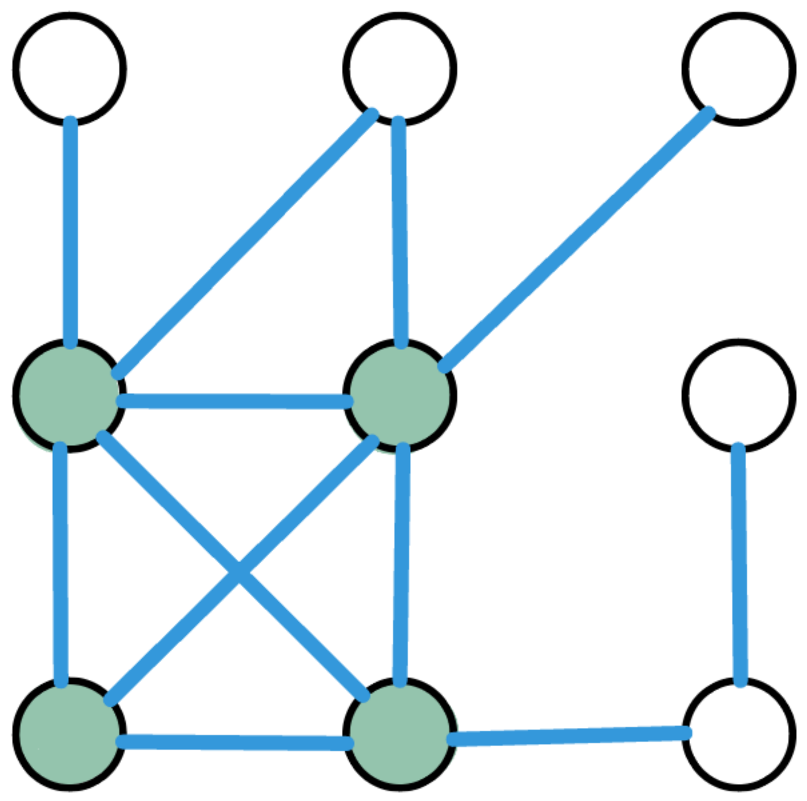

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Extremal Combinatorics

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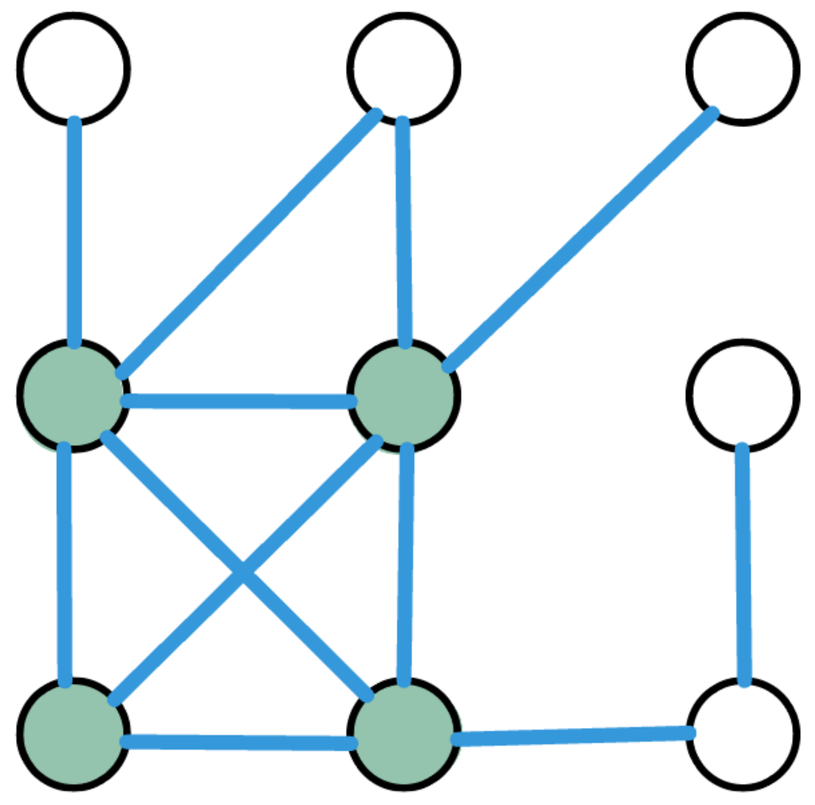

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Conjecture [GP'17]

$\frac{n}{2}$-RAMSEY $\in$ PPP

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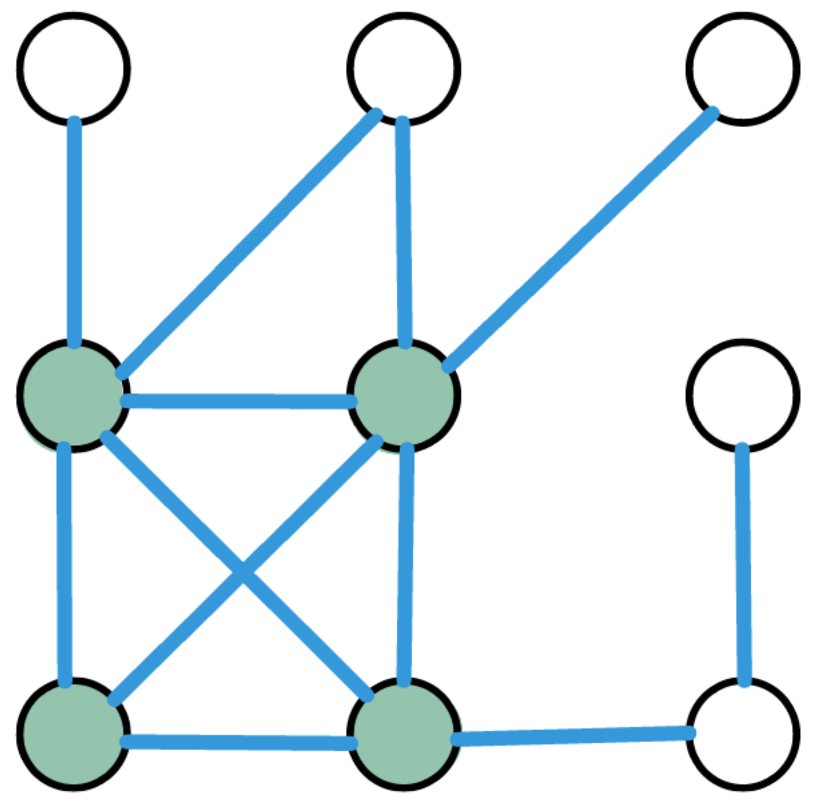

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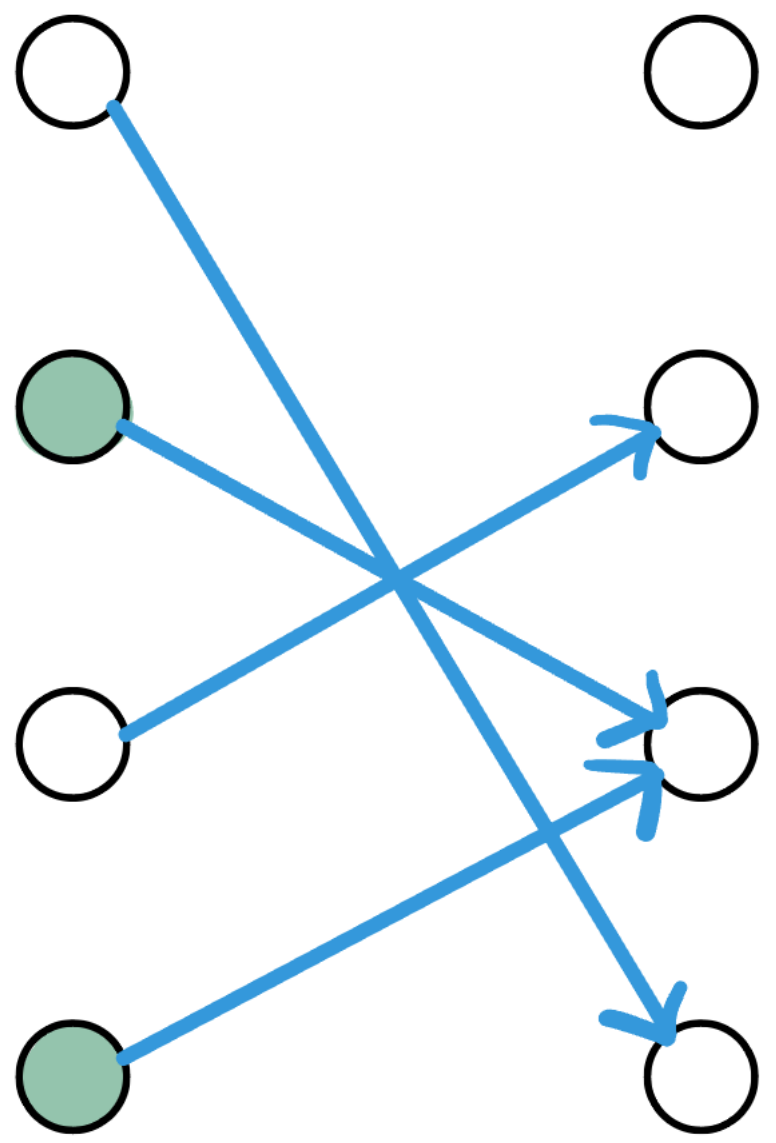

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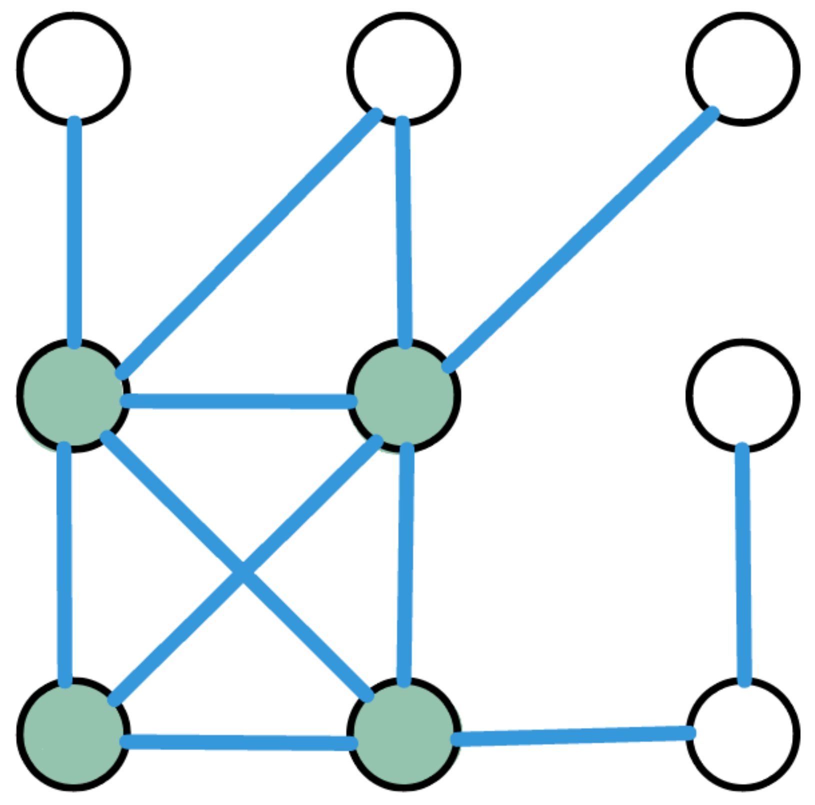

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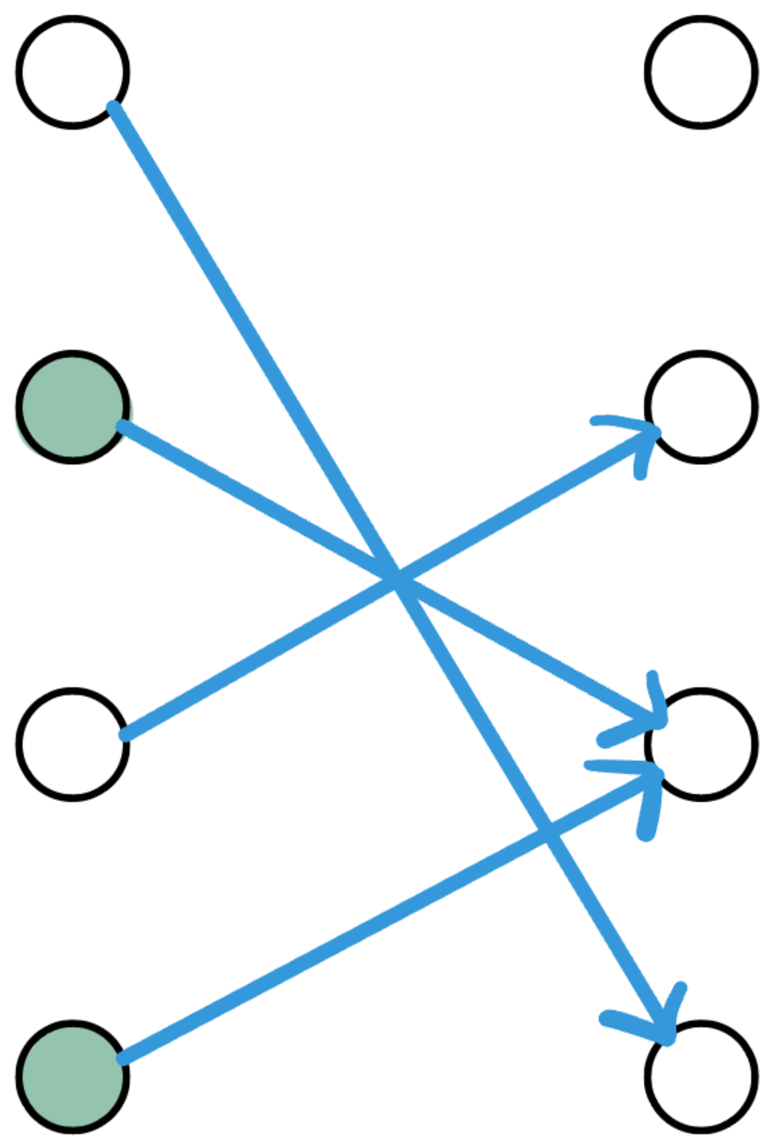

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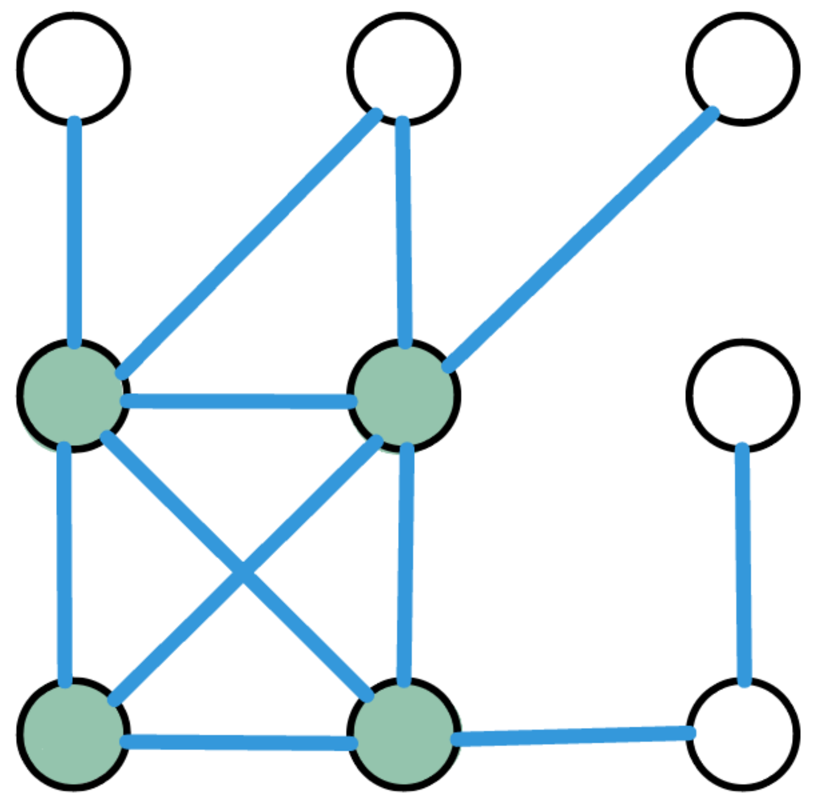

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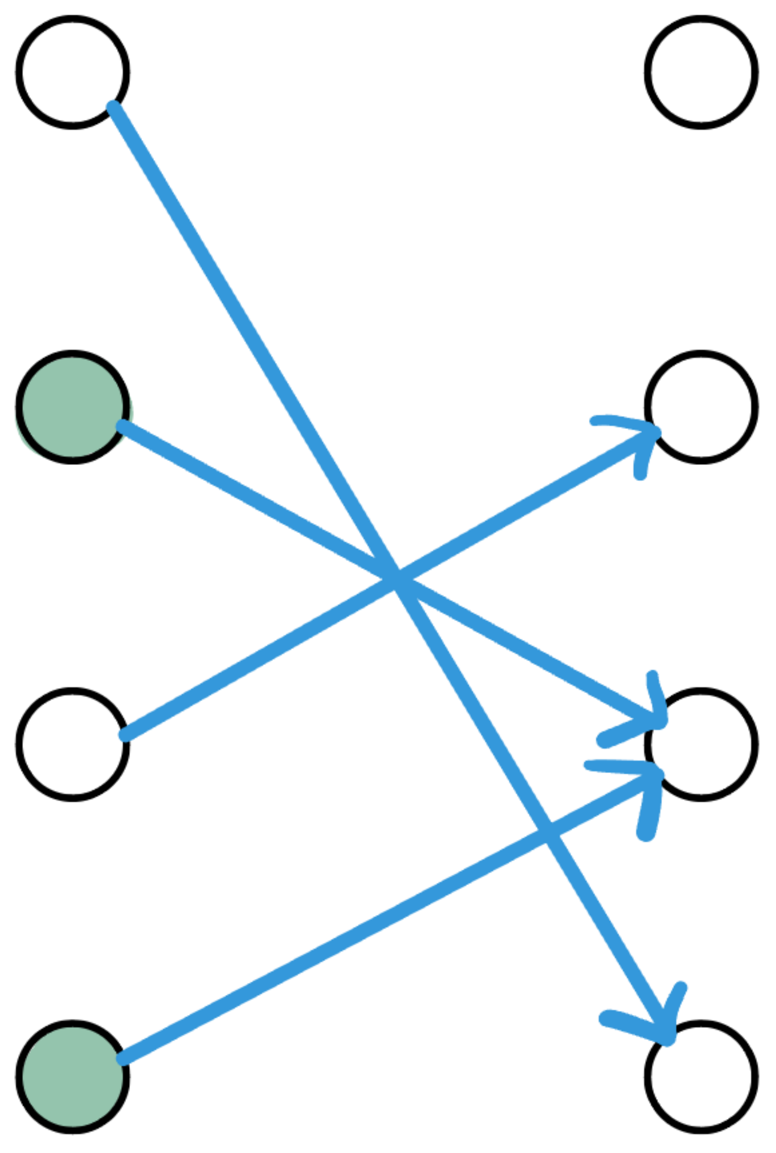
RAMSEY ~~∈~~ PPP

FALSE (BLACK-BOX)

OUR WORK



$N = 2^n$

Generalized Pigeonhole Principle

t-PIGEON$_N$

Input $[M] \to [N]$ $\quad (M = (t-1)N)$

Solutions $\to$ $t-1$ pigeons mapped to 0

$\to$ t collision

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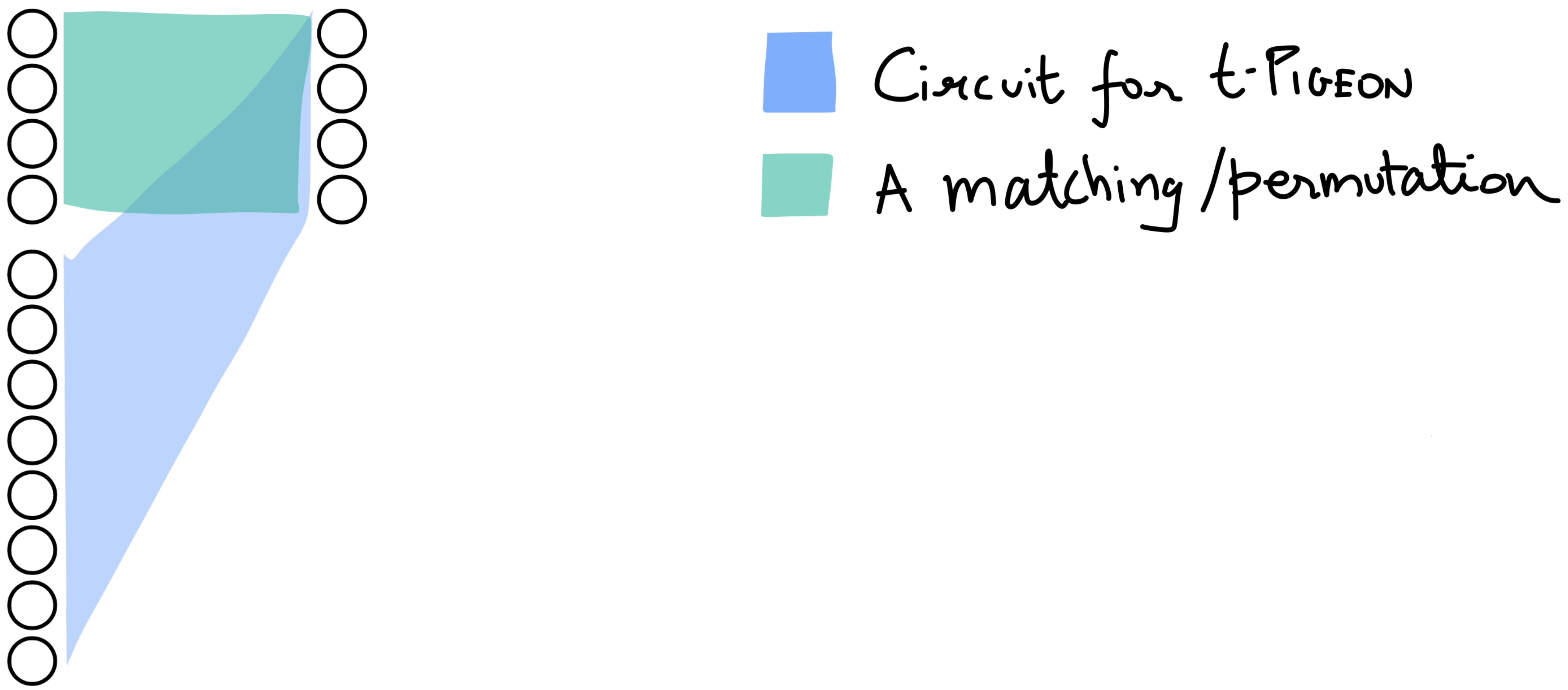
Proof



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Polynomial Averaging Principle (PAP)

Everything reducible to
n-PIGEON$_N$

Lemma For any $t(n)$, t-PIGEON$_N \leq (t+1)$-PIGEON$_N$

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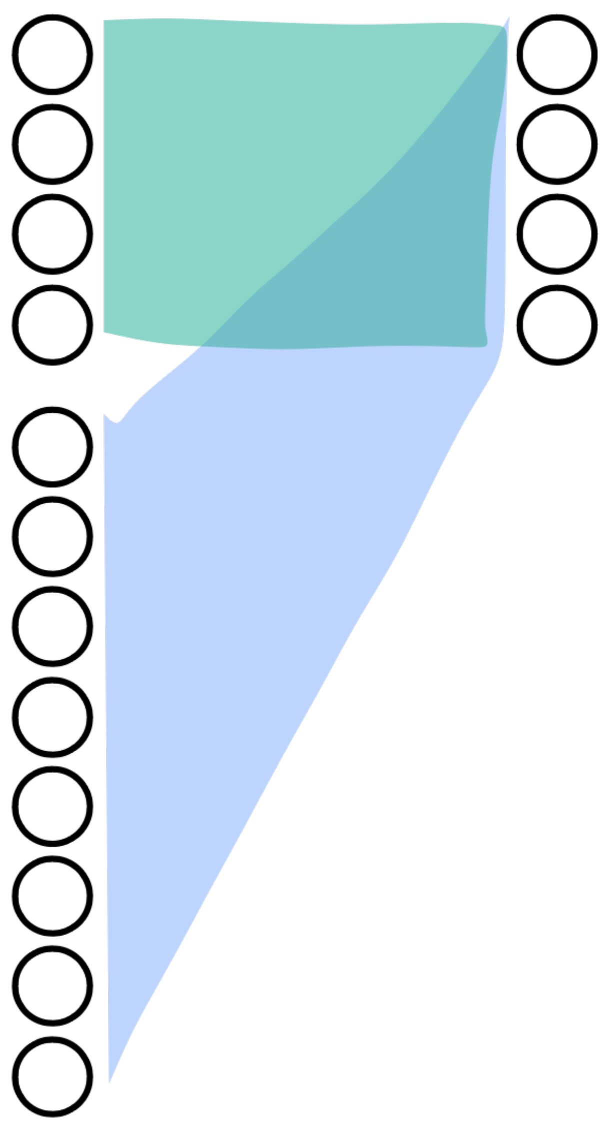

Circuit for t-PIGEON

A matching/permutation

$N = 2^n$

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equivalent to poly(n)-PIGEON$_N$

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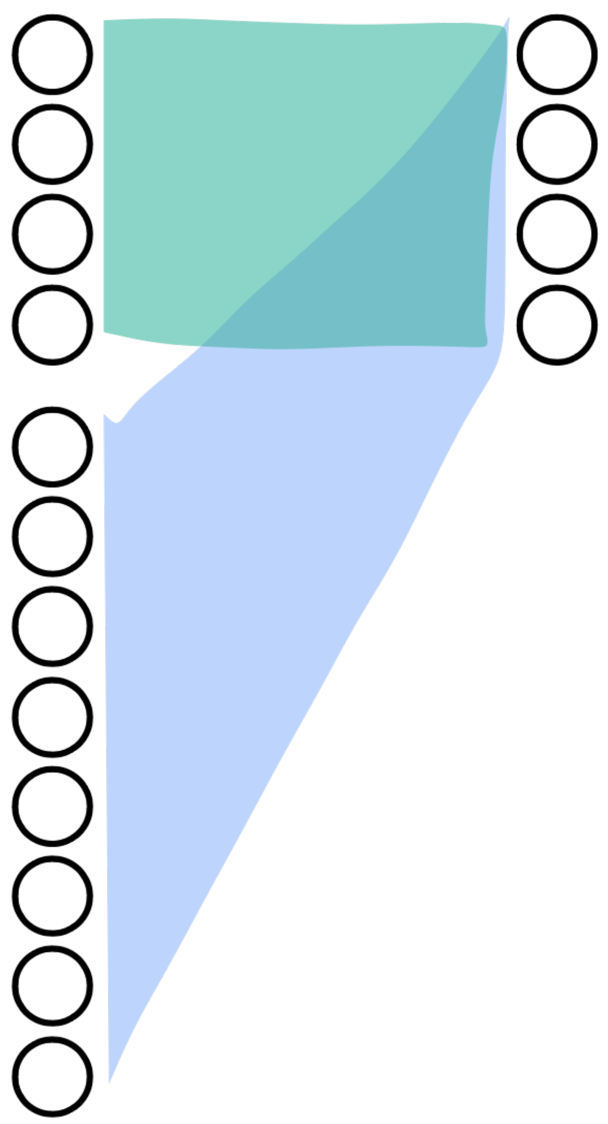

Circuit for t-PIGEON

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~~Generalized Pigeonhole Principle~~

Pecking Order

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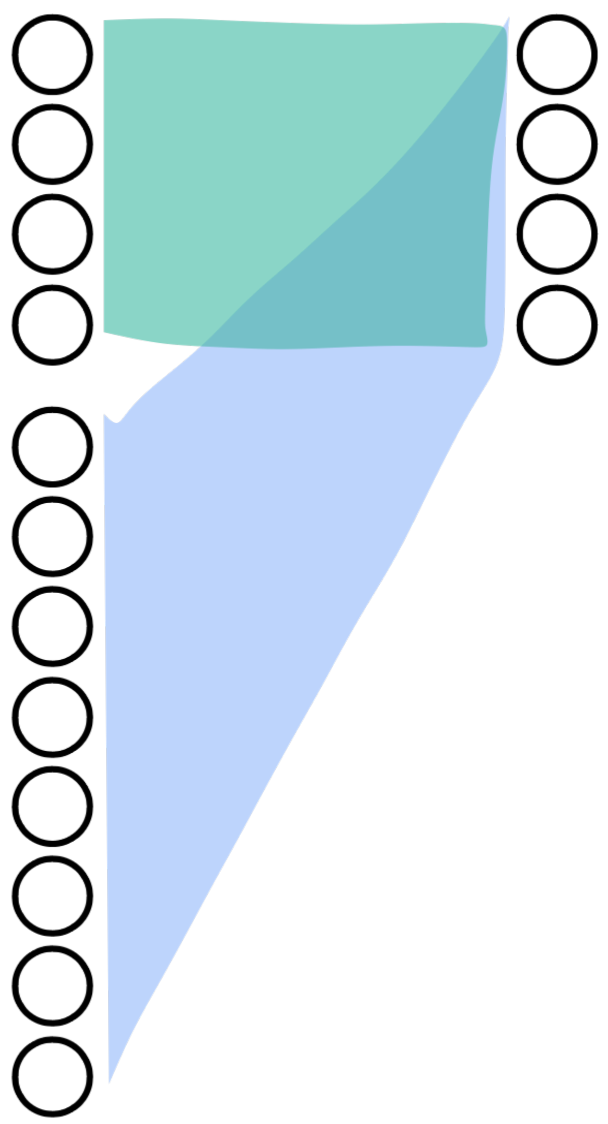

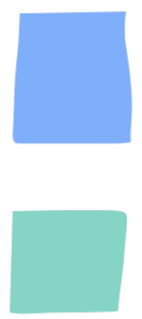

A Reduction

$N = 2^n$
$M = 2^m$

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A Reduction

Theorem When $2t(2n-1) \le m$, $t\text{-PIGEON}_N^M \le \text{RAMSEY}_M$

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Proof $h : [M] \to [N]$

G_0 on N vertices with no C/IS of size $2n$

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Define $G = G_0 \otimes h$

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Define $G = G_0 \otimes h$ graph hash product [KNY17]

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$$(u,v) \in E \begin{cases} h(u) = h(v) \\ (h(u), h(v)) \in E_0 \end{cases}$$

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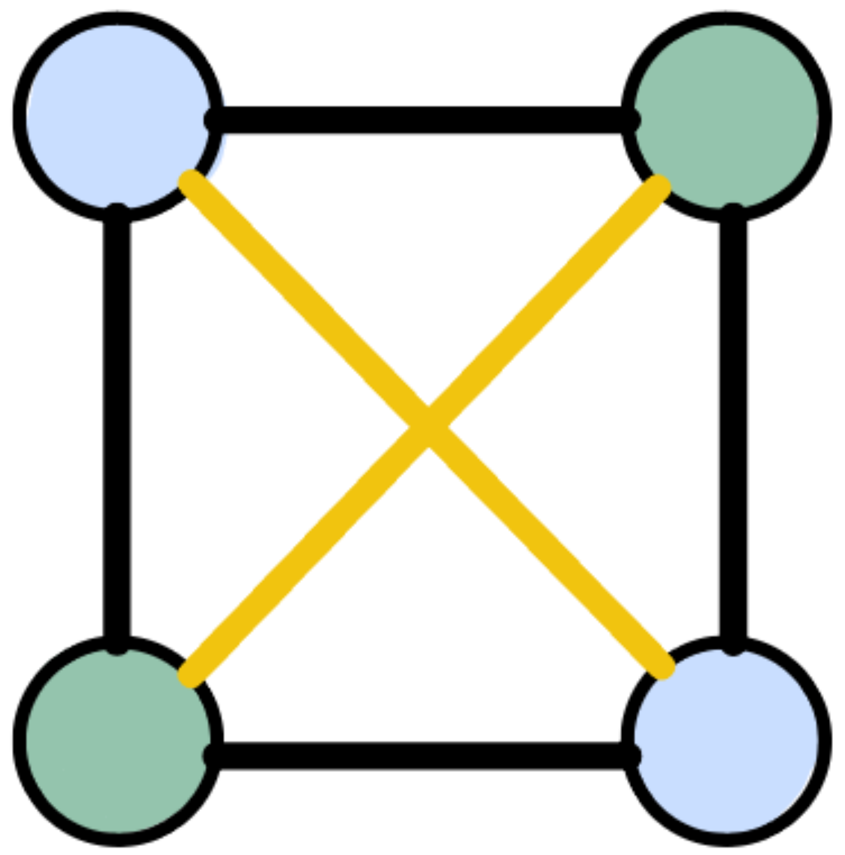

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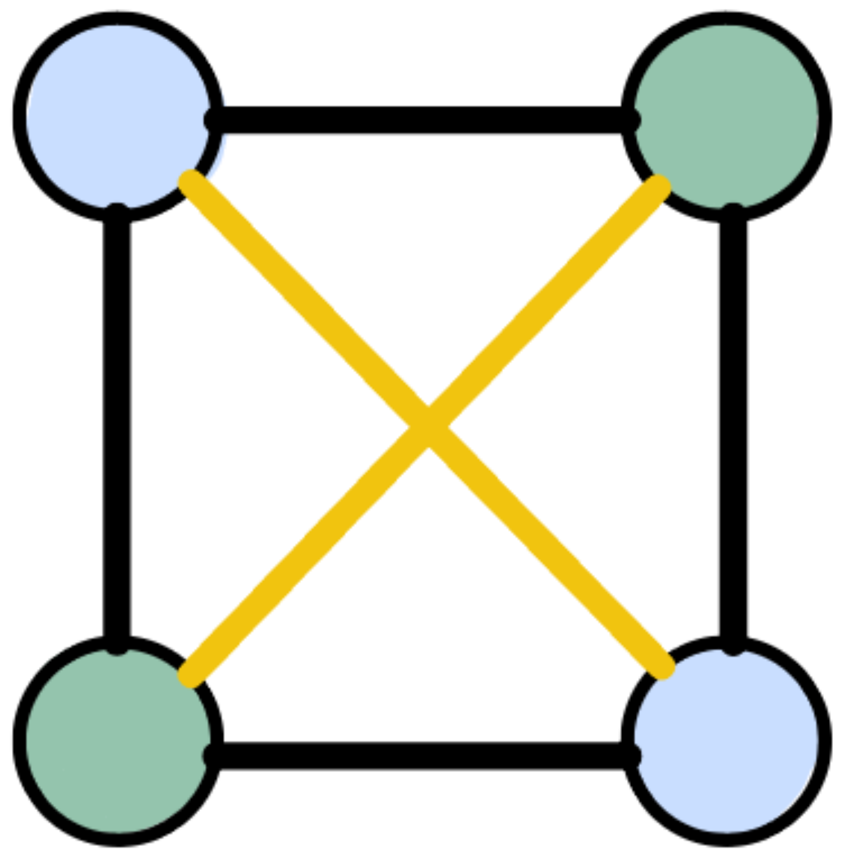
$$(u,v) \in E \begin{cases} h(u) = h(v) \\ (h(u), h(v)) \in E_0 \end{cases}$$

$\frac{m}{2}$-clique in G has at most $2n-1$ distinct vertices $h(u)$



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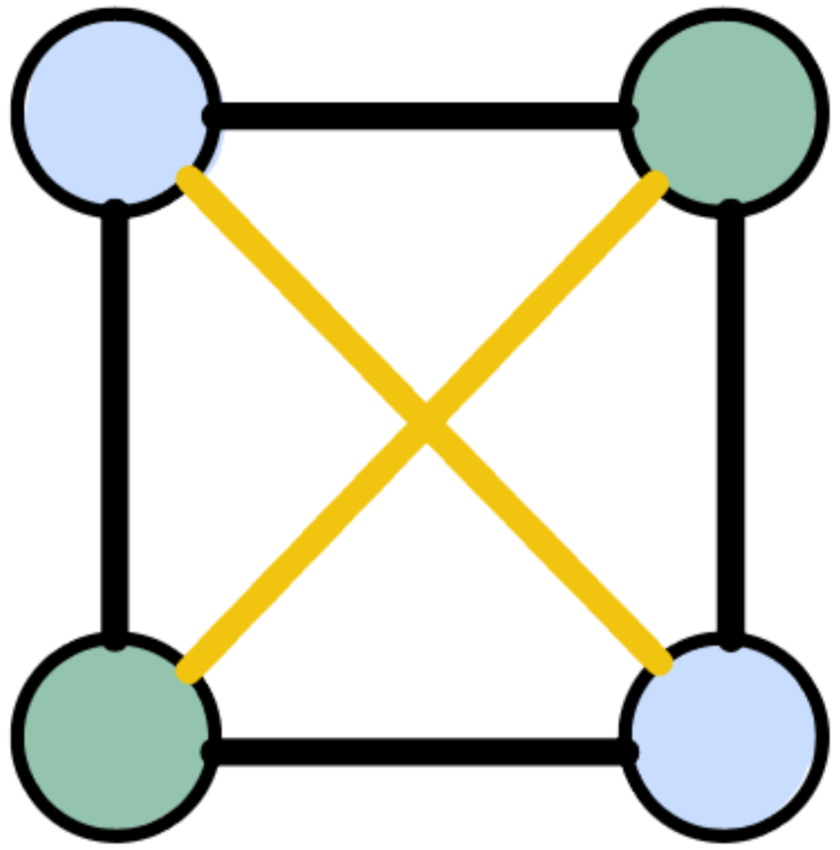

$\therefore$ must contain a $t = \frac{m}{2(2n-1)}$ collision

A Seperation

$N = 2^n$
$M = 2^m$

RAMSEY

|

n-PIGEON$^{N^{n^2}}_{N}$ $\dashrightarrow$ SAP

$N = 2^n$

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A Seperation

Theorem $n\text{-PIGEON}^{M}_{N} \not\preceq_{dt} n^{o(1)}\text{-PIGEON}_{N}$ whenever $m = \mathrm{poly}(n)$

RAMSEY

$|$

$n\text{-PIGEON}^{N^{n^2}}_{N}$

SAP

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By the reduction,

$$\begin{array}{ccc} \text{RAMSEY} & & \text{SAP} \\ | & \nearrow & \\ n\text{-PIGEON}_N^{N^{n^2}} & & \end{array}$$

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RAMSEY SAP

$n\text{-PIGEON}_N^{N^{n^2}}$

Corollary

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SAP

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whenever $m = \mathrm{poly}(n)$

By the reduction,

Corollary $\textsc{Ramsey}_N \not\leq_{dt} n^{o(1)}\text{-}\textsc{Pigeon}_N$!!

Subpolynomial Averaging Principle (SAP)

$$\begin{array}{ccc} \textsc{Ramsey} & & \textsc{SAP} \\ | & \nearrow & \\ n\text{-}\textsc{Pigeon}_N^{N^{n^2}} & & \end{array}$$

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Subpolynomial Averaging Principle (SAP)

$$\begin{array}{ccc} \text{RAMSEY} & & \text{SAP} \\ | & \nearrow & \\ n\text{-PIGEON}_N^{N^{n^2}} & & \end{array}$$

Most of the work goes into proving the Theorem.

The Full Picture

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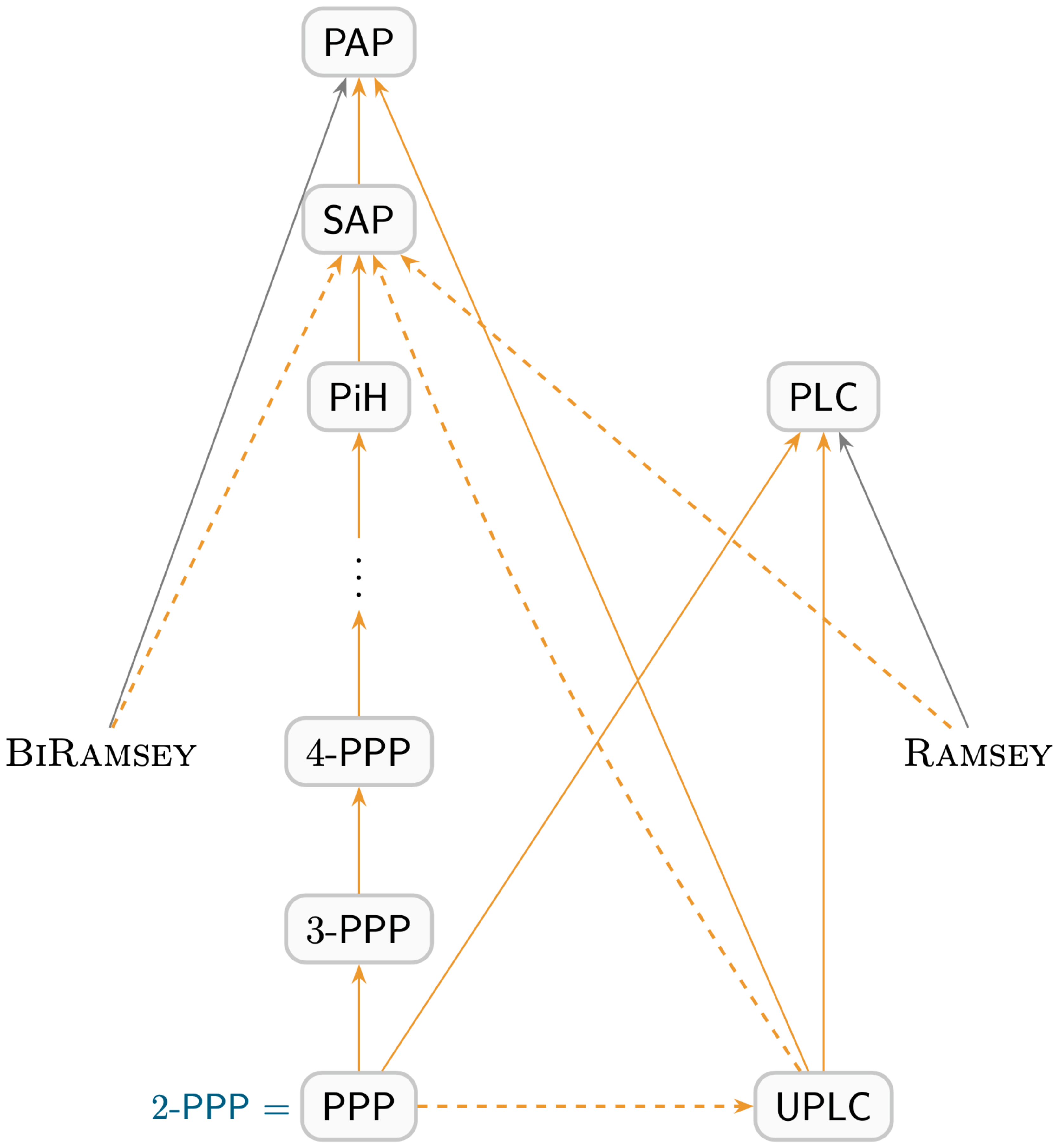

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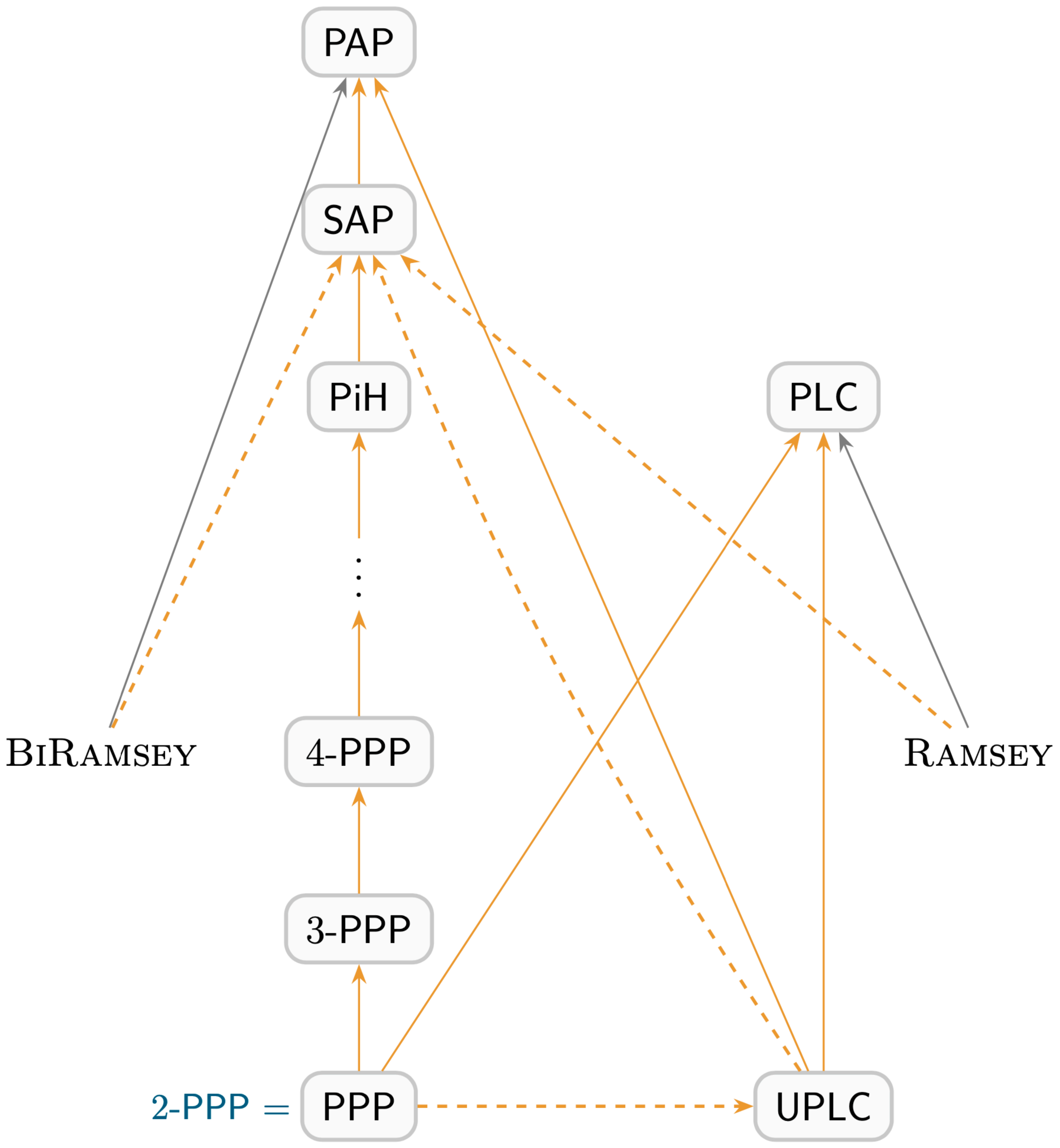

PiH Pigeon Hierarchy

$$= \bigcup_{t=2}^{\infty} t\text{-PPP}$$

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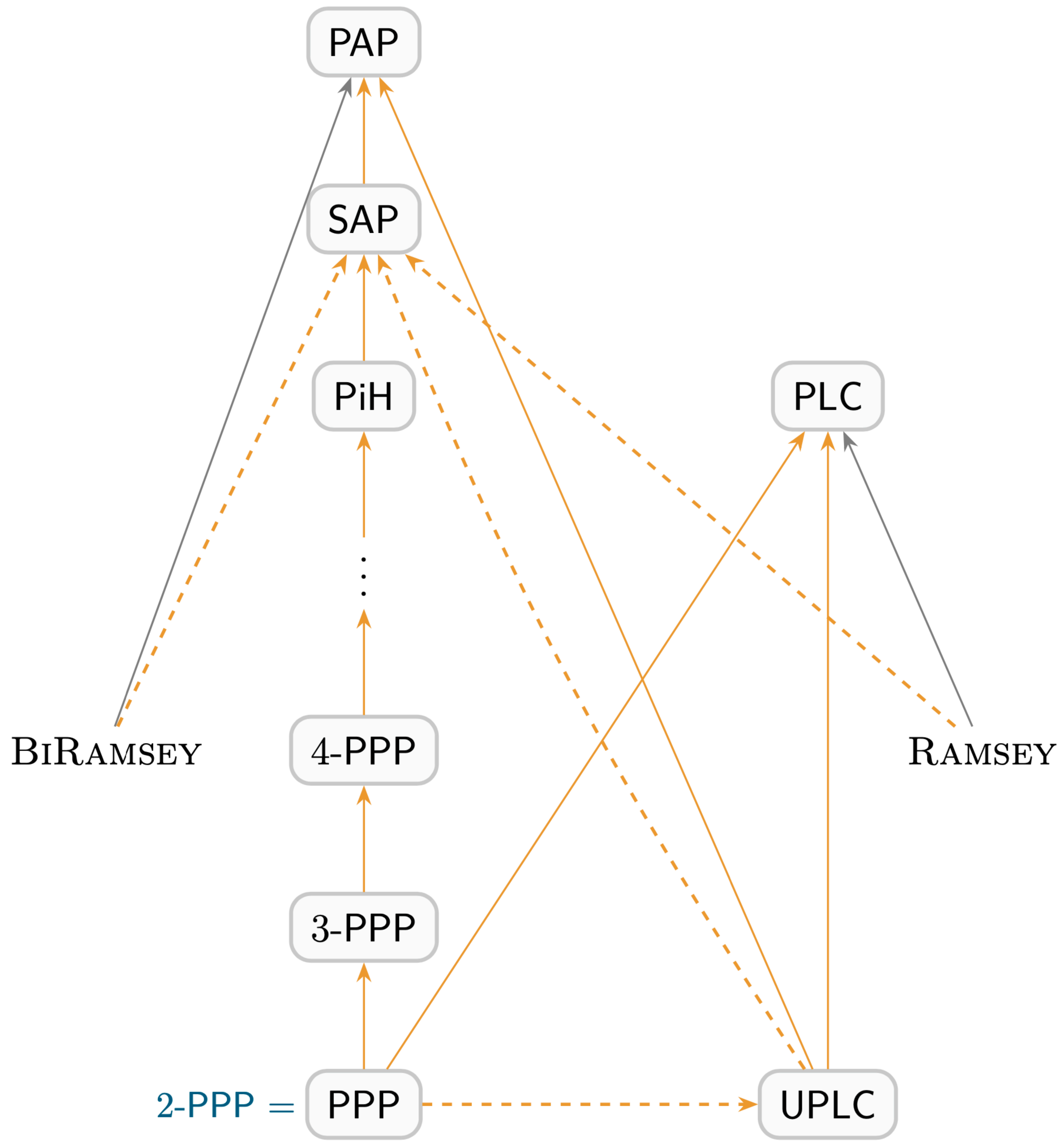

PiH Pigeon Hierarchy

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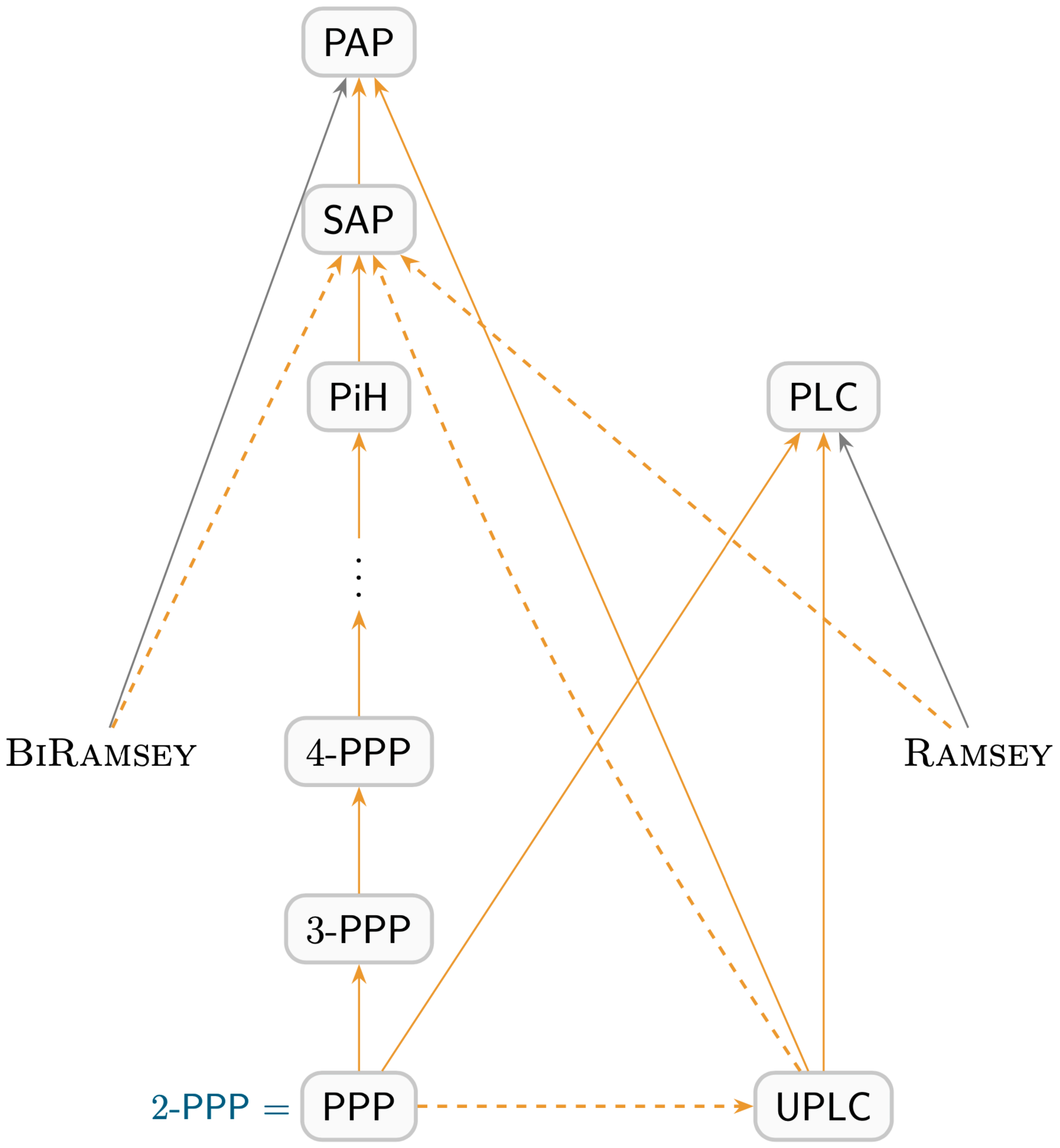
$A \dashrightarrow B$

$A \not\subseteq_{dt} B$

$A \longrightarrow B$

$A \subsetneq_{dt} B$

The Full Picture



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Everything in orange is

Our Work

Iterated Pigeonhole

$N = 2^n$

UPLC Given $P : \{0,1\}^n \to \{0,1\}^{n-1}$
find a lower-triangular collision.

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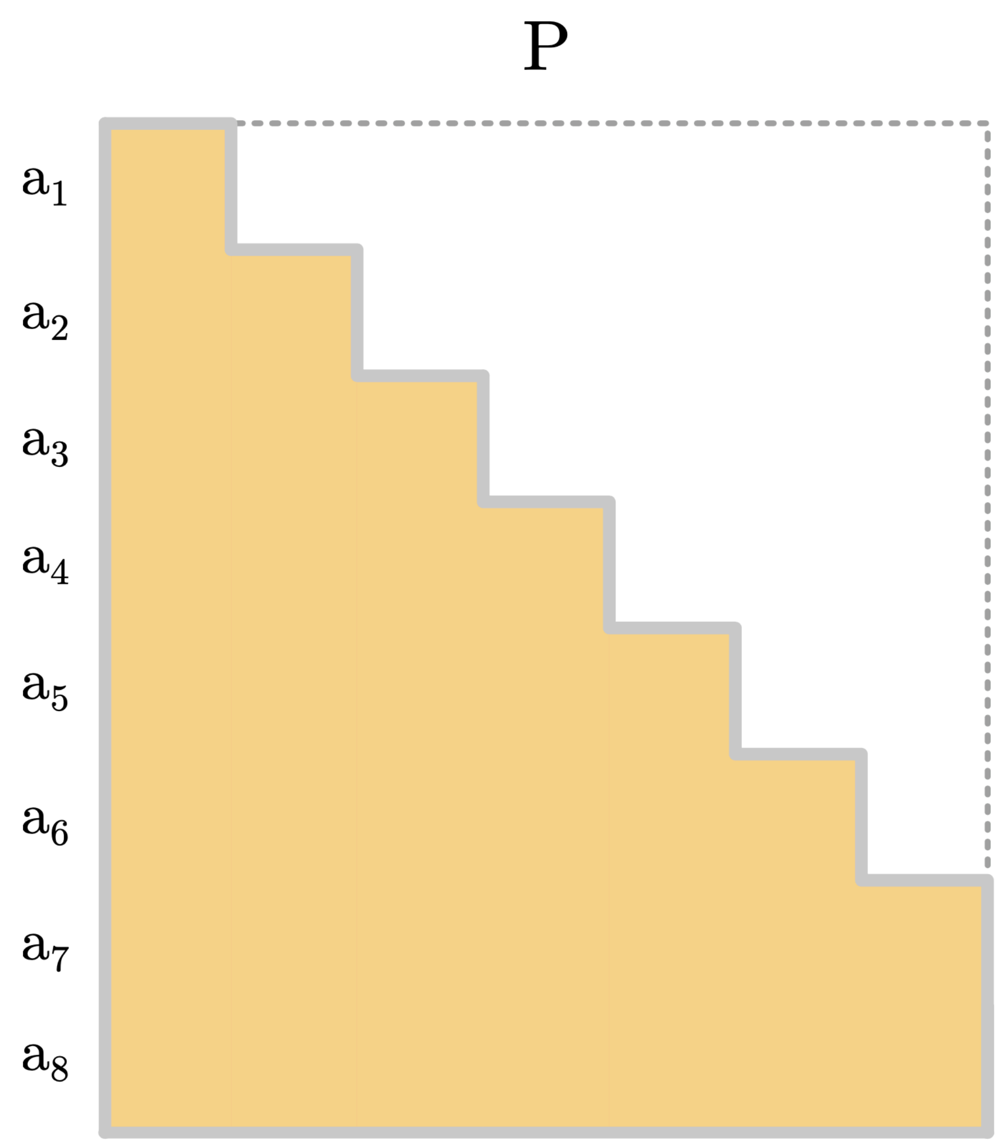

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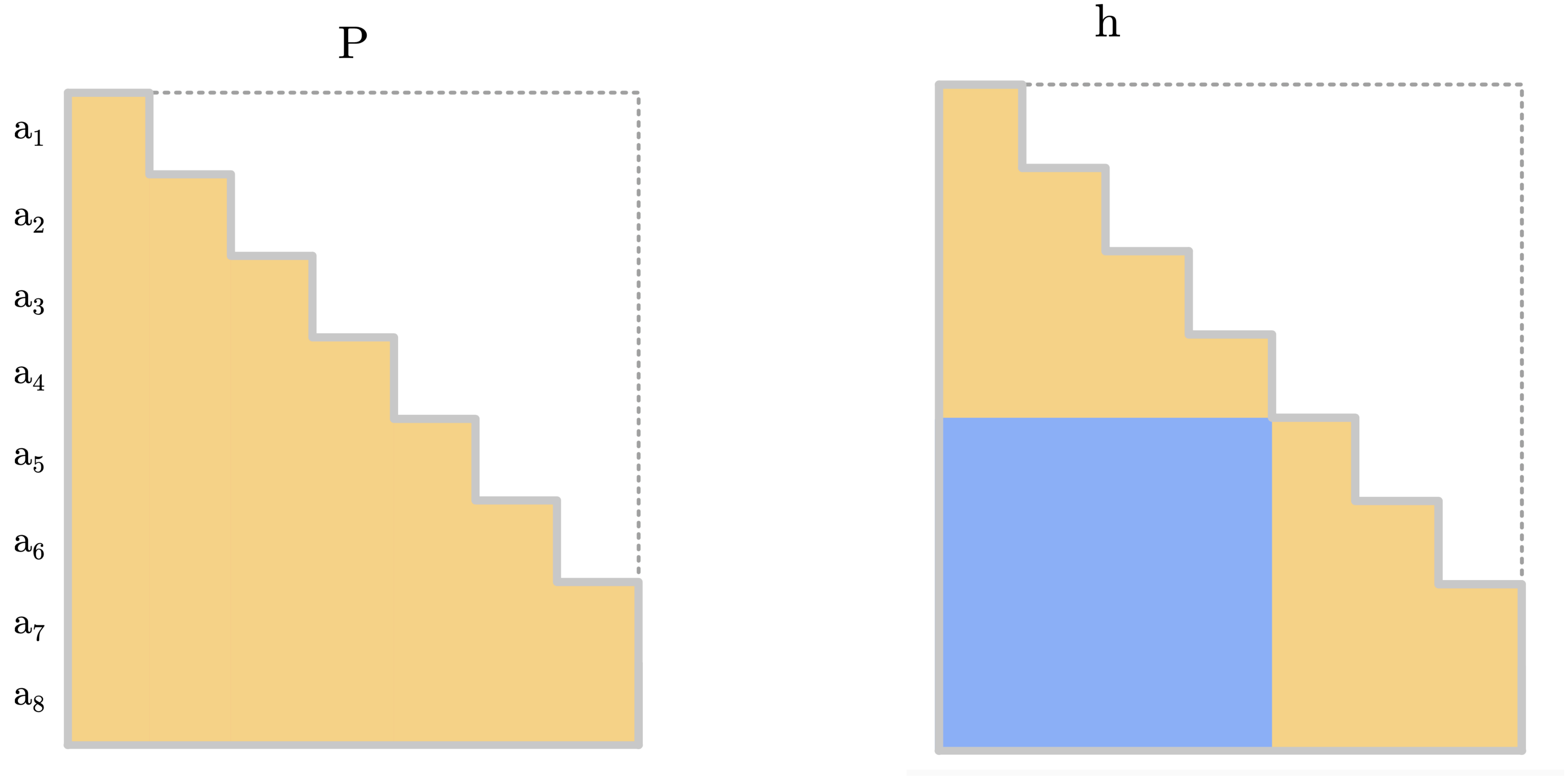

$\frac{n}{2}\text{-PIGEON}^{N}_{\sqrt{N}} \leq \text{UPLC}$

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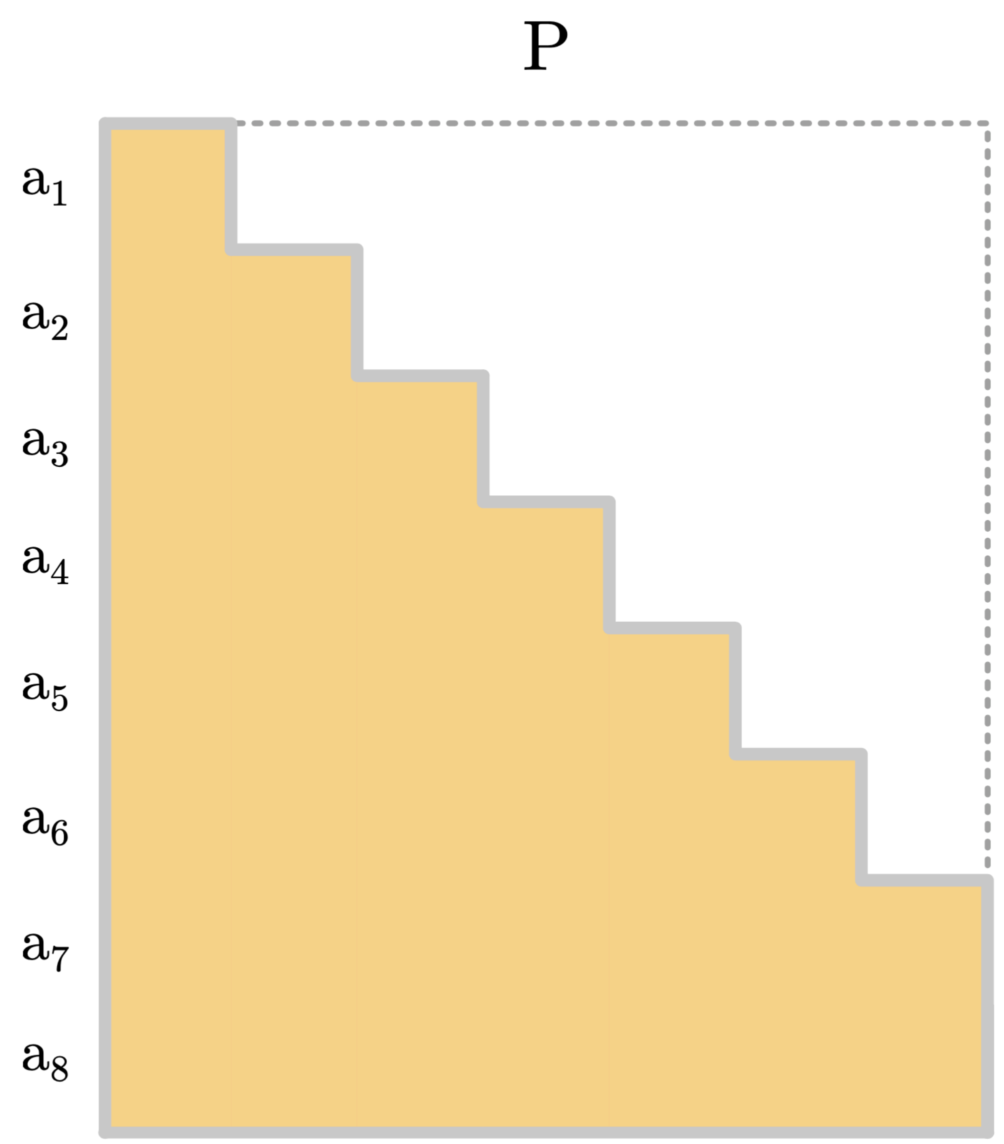

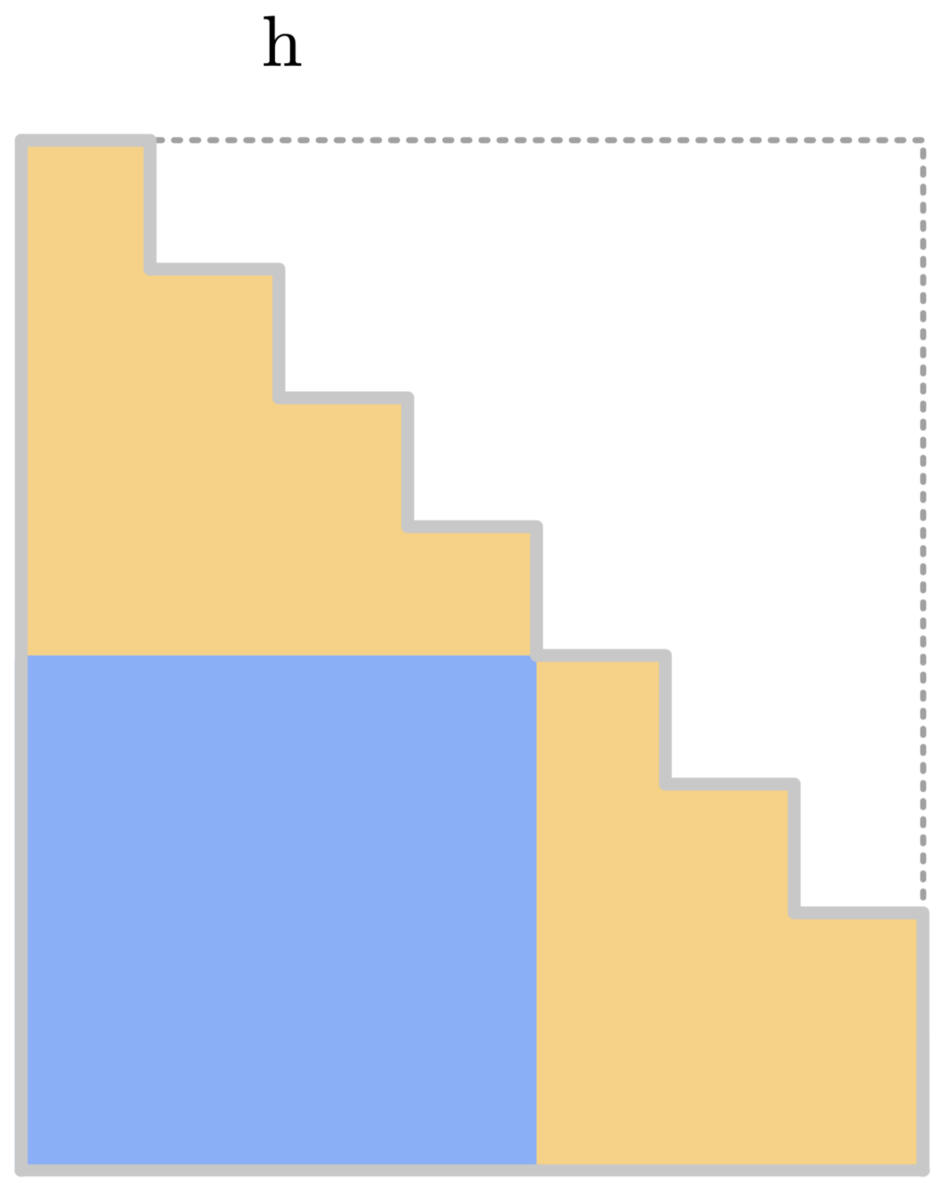

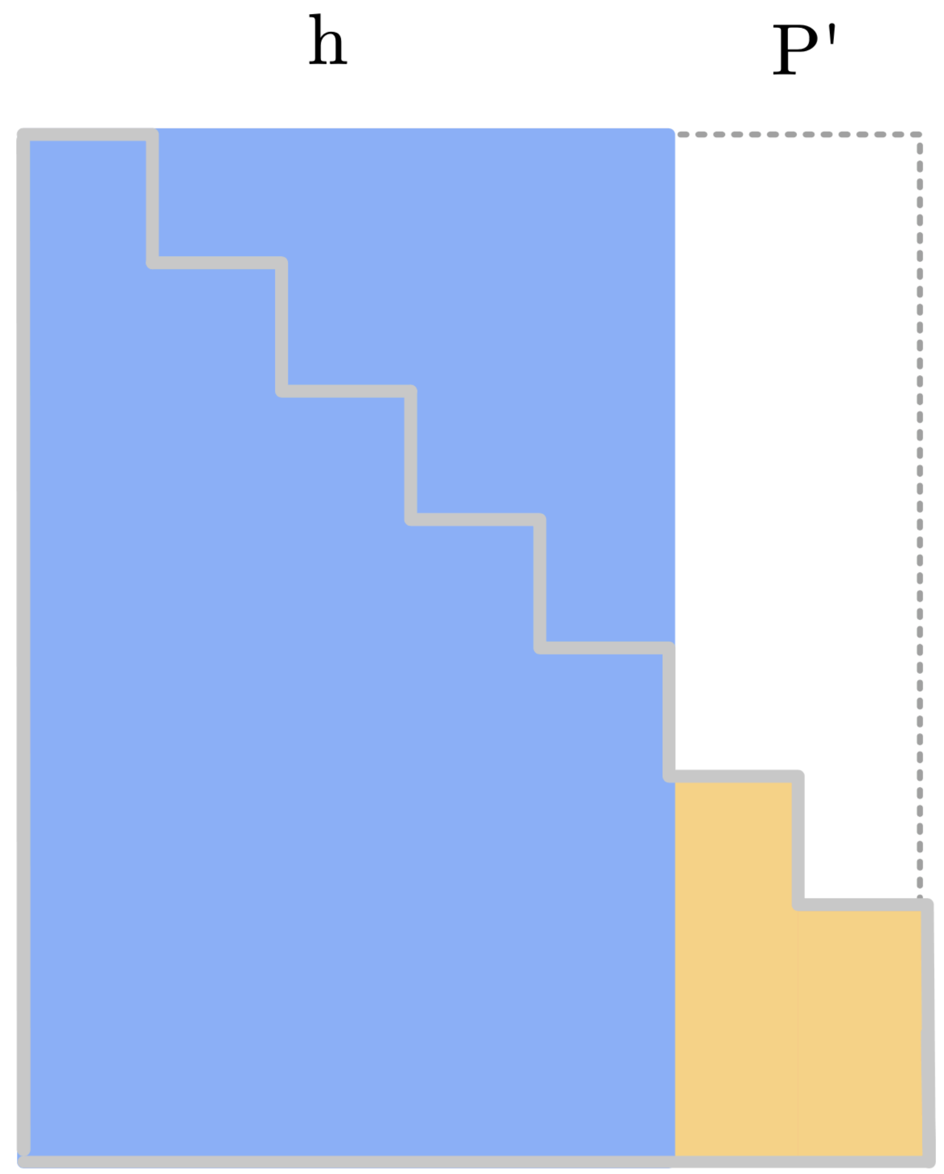

$\frac{n}{2}\text{-PIGEON}^{N}_{\sqrt{N}} \leq \text{UPLC}$

$\text{UPLC} \leq n\text{-PIGEON}^{N}_{N/n}$

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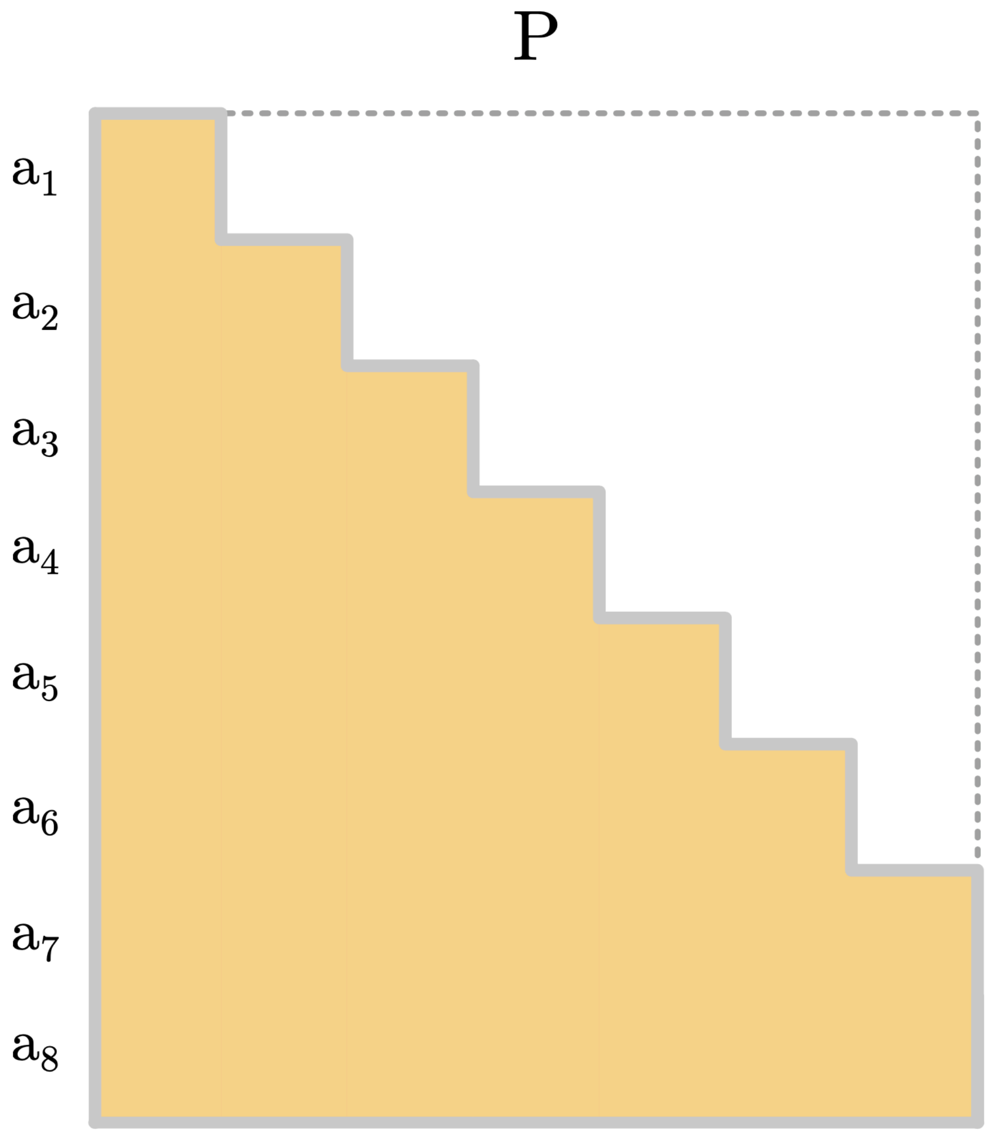
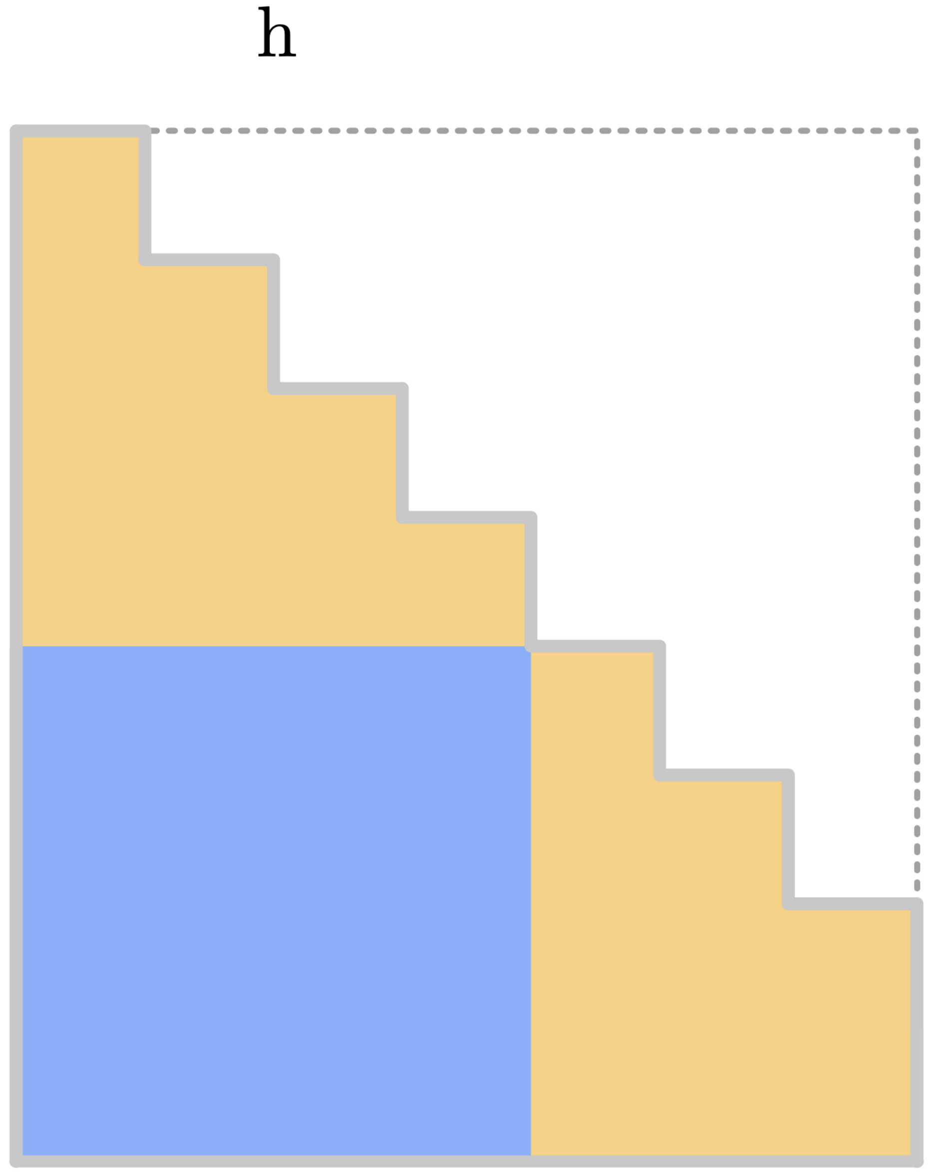
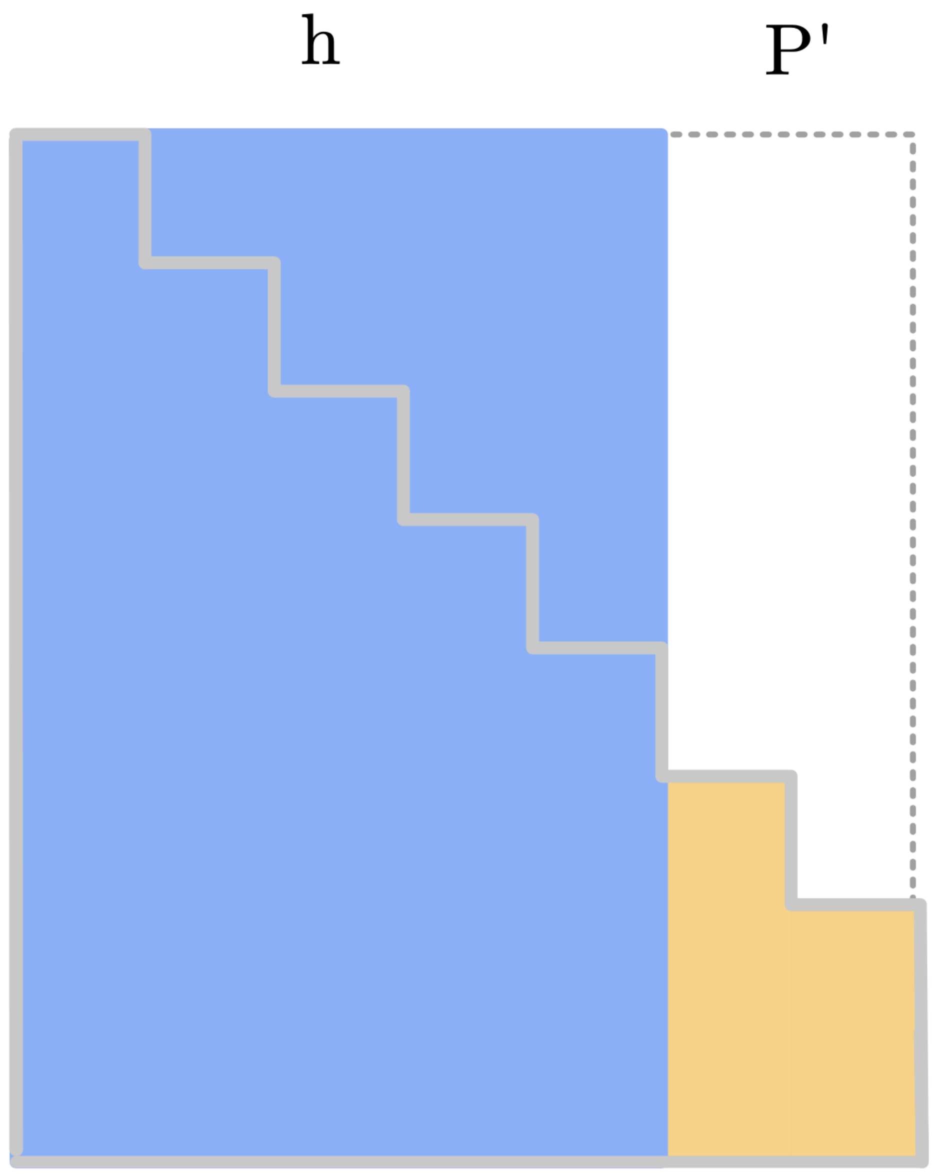




$\frac{n}{2}$-$\mathrm{PIGEON}^{N}_{\sqrt{N}} \leq \mathrm{UPLC}$



$\mathrm{UPLC} \leq n$-$\mathrm{PIGEON}^{N}_{N/n}$

PLC output of P is given bit-by-bit adaptively

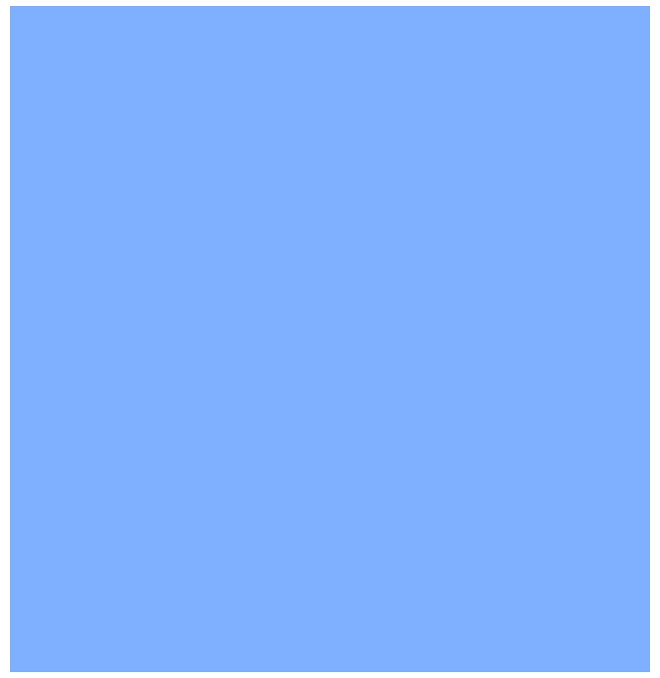
Ramsey vs Bipartite Ramsey

Ramsey vs Bipartite Ramsey

Adjacency matrix view

Ramsey vs Bipartite Ramsey

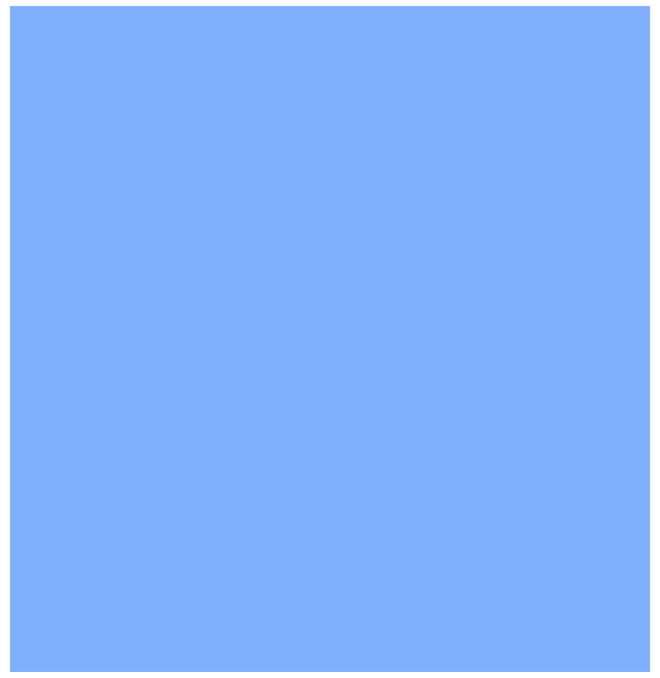
Adjacency matrix view



Ramsey vs Bipartite Ramsey

Adjacency matrix view

Bipartite



monochromatic square

Ramsey vs Bipartite Ramsey

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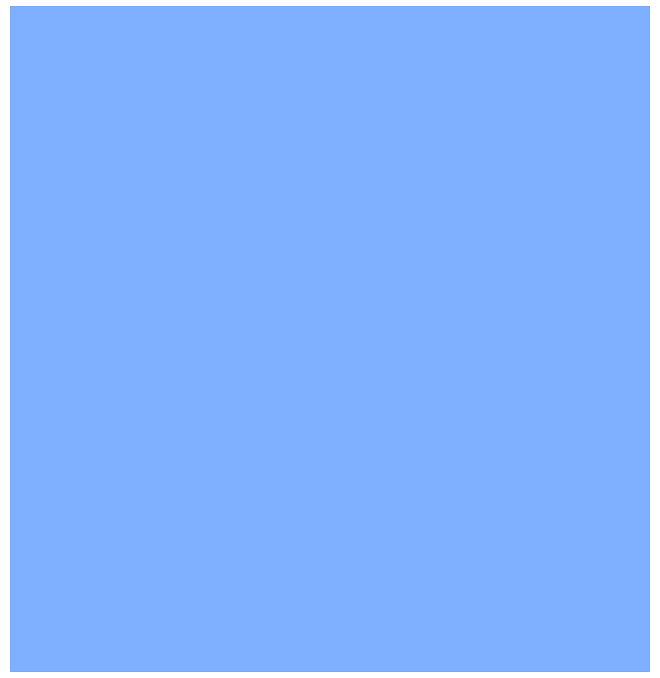

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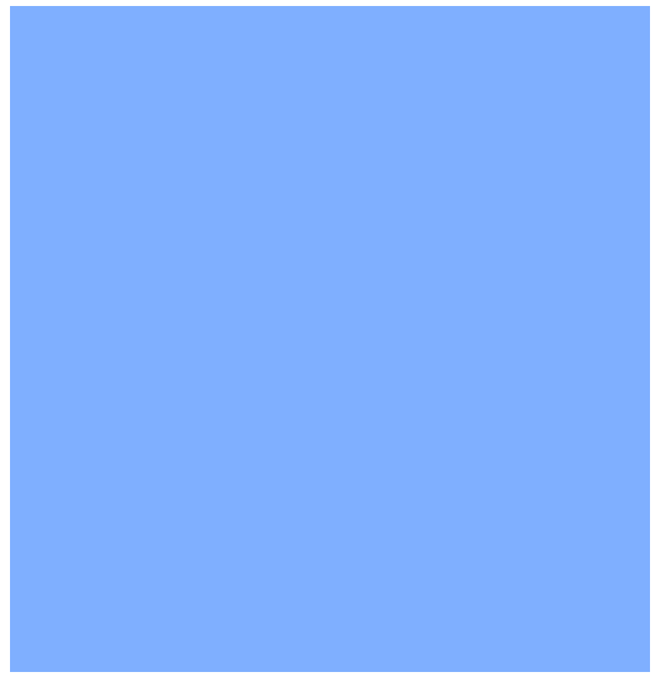

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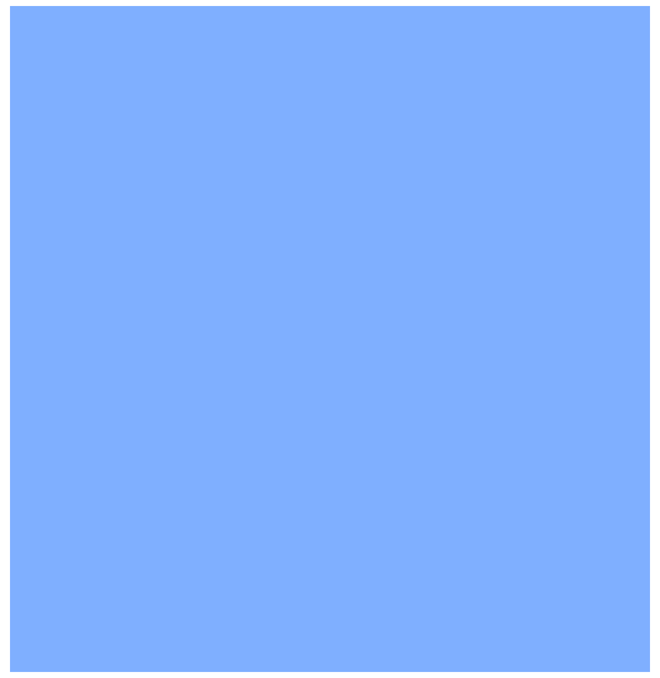

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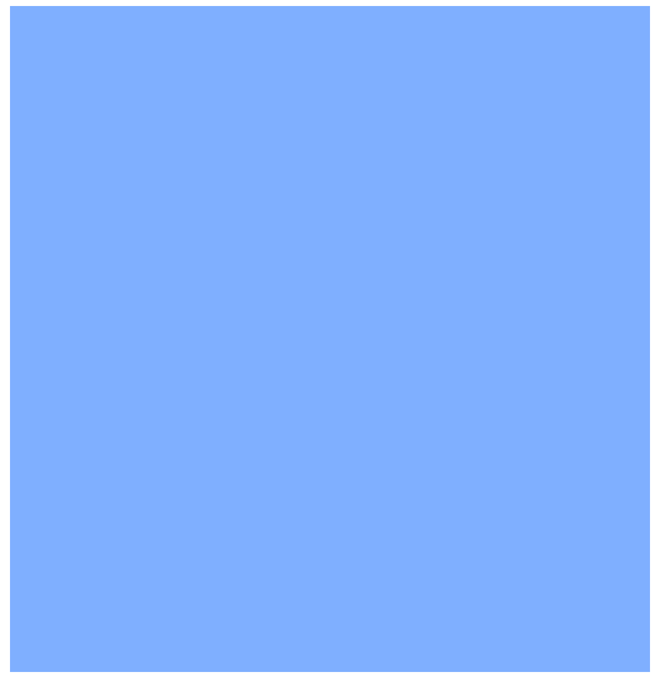

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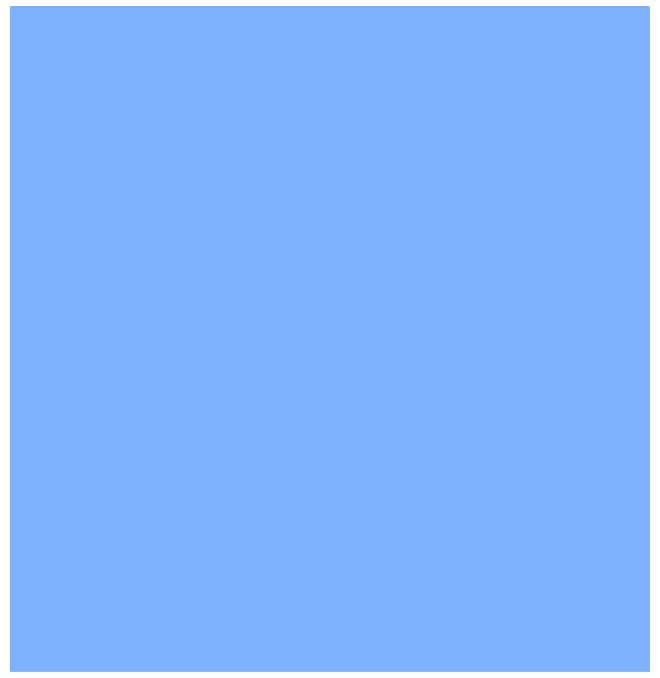

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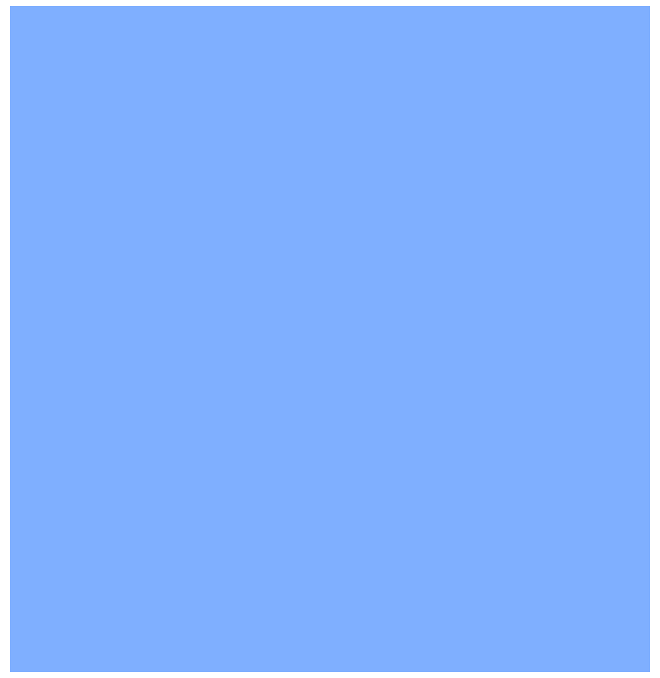

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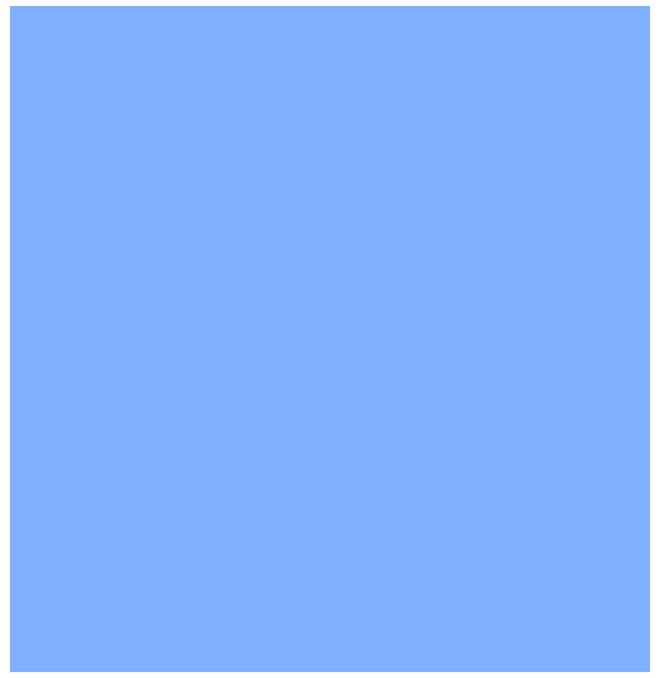

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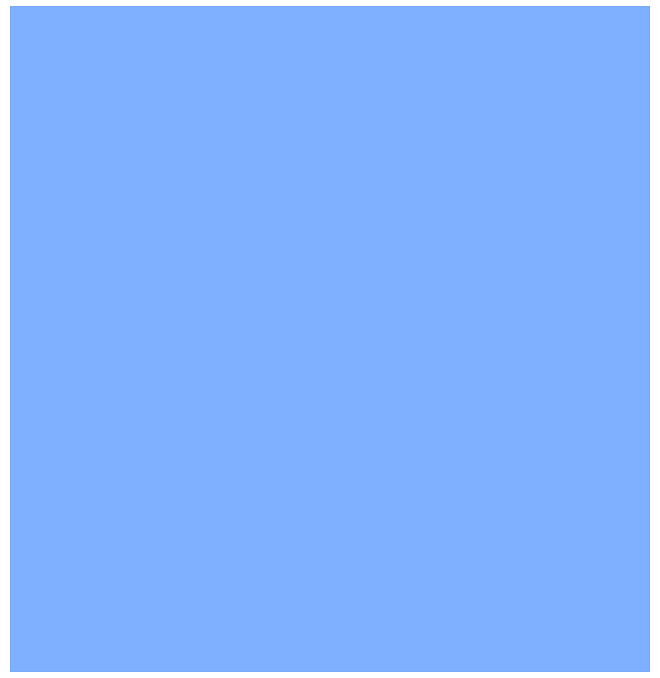

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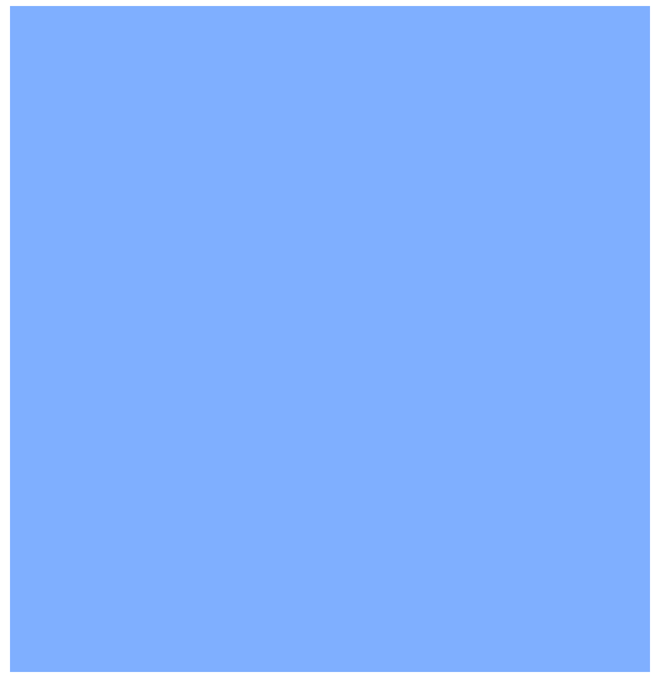

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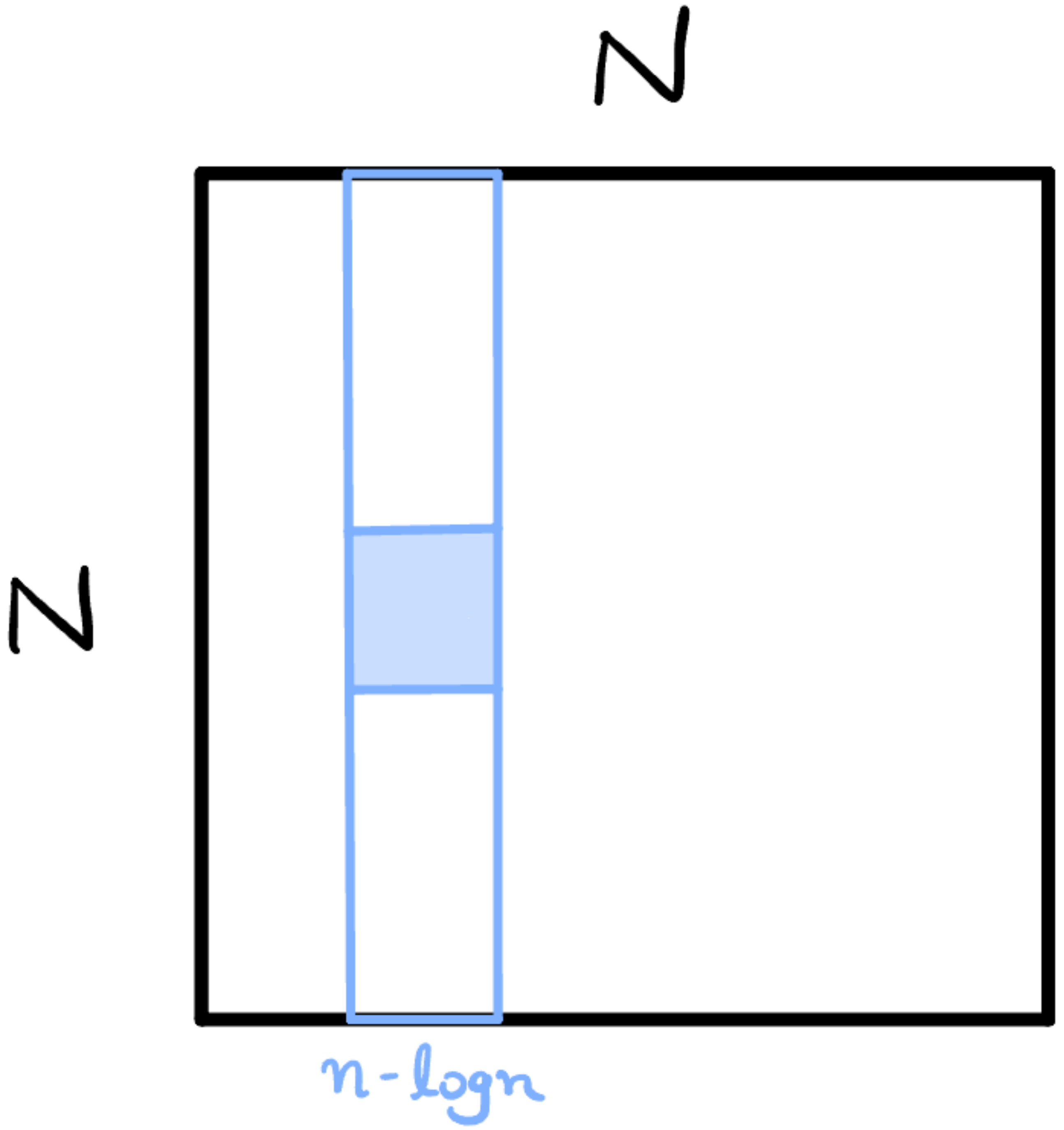

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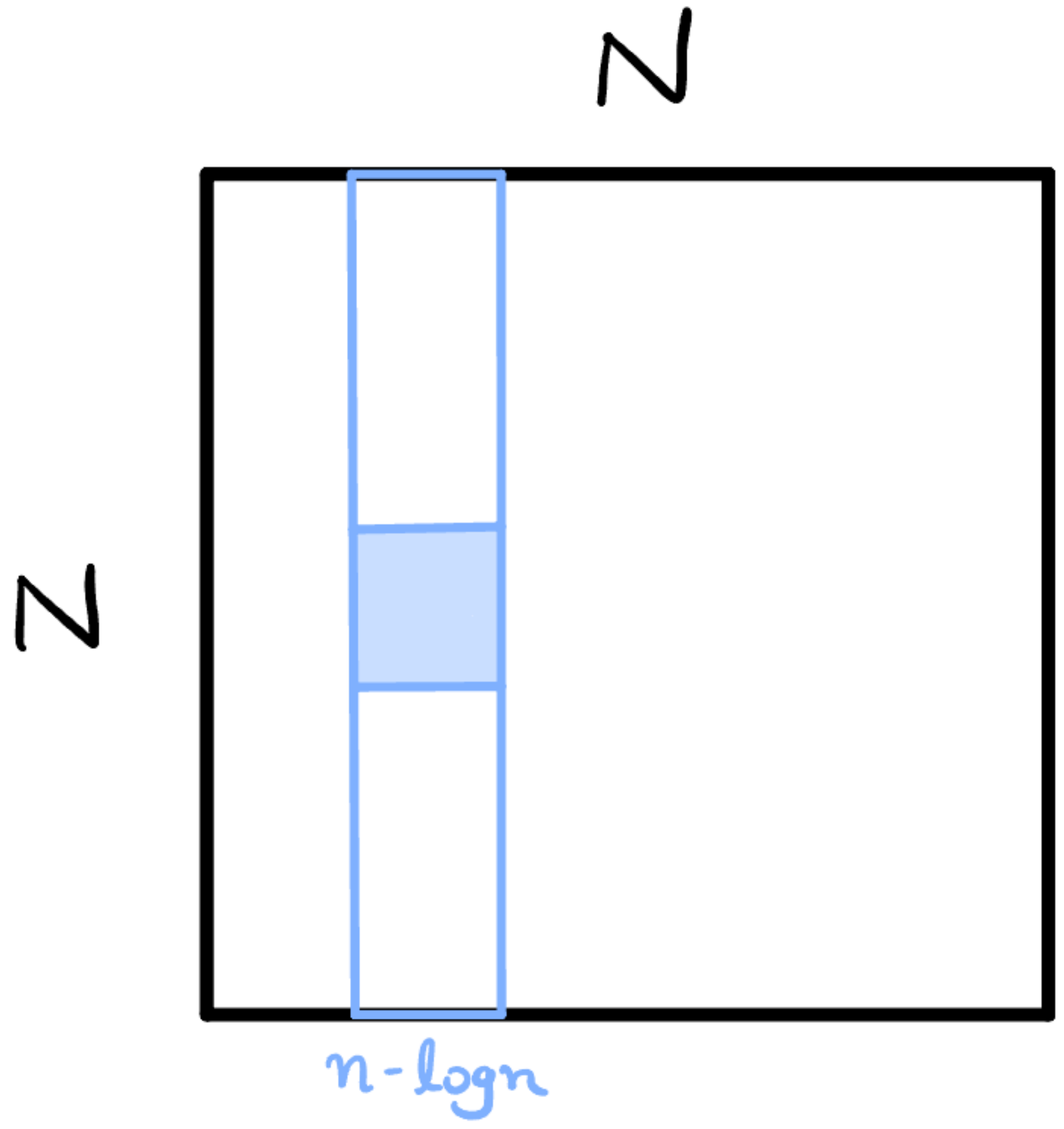

PIGEONS vertices in left partition

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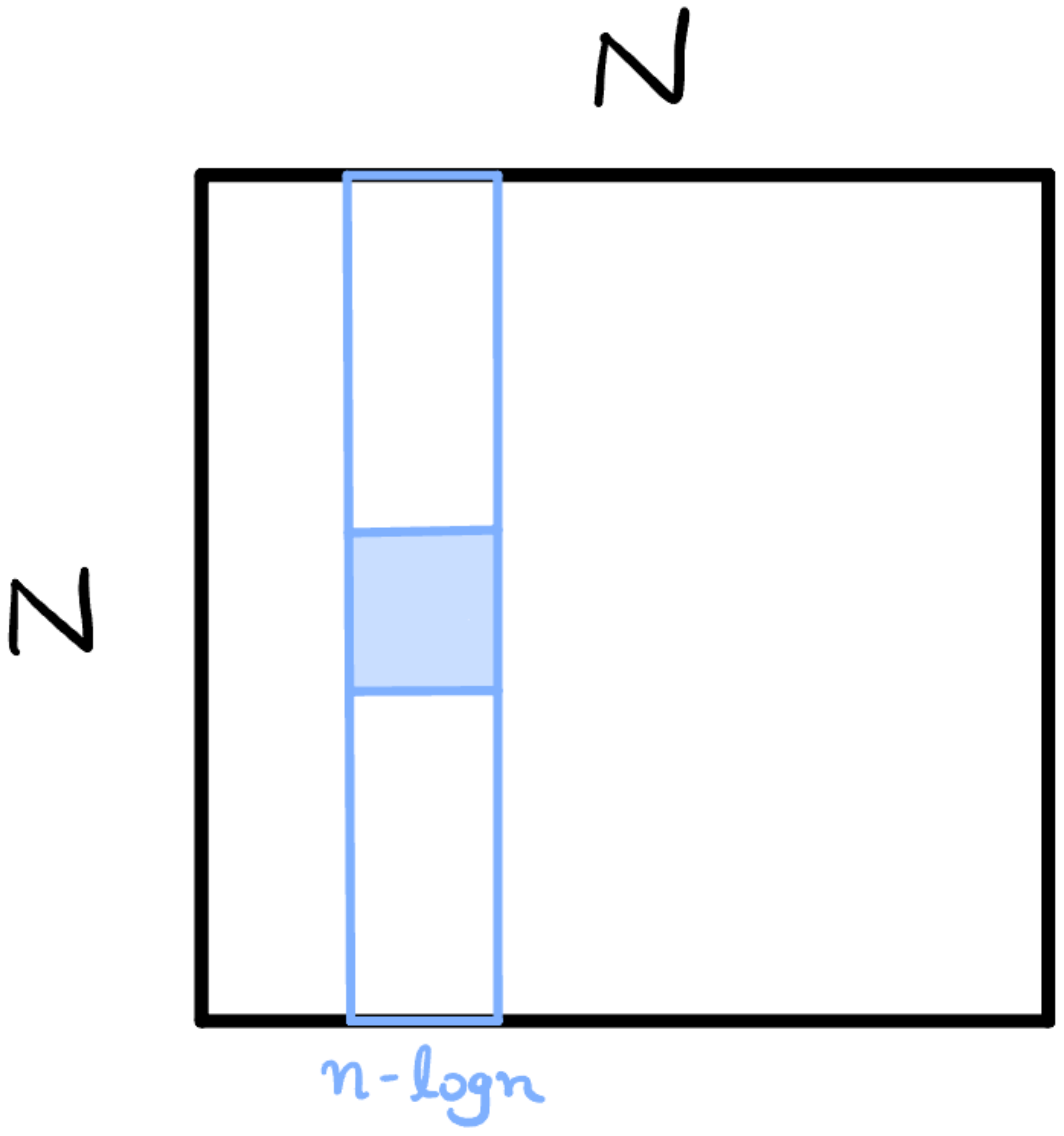

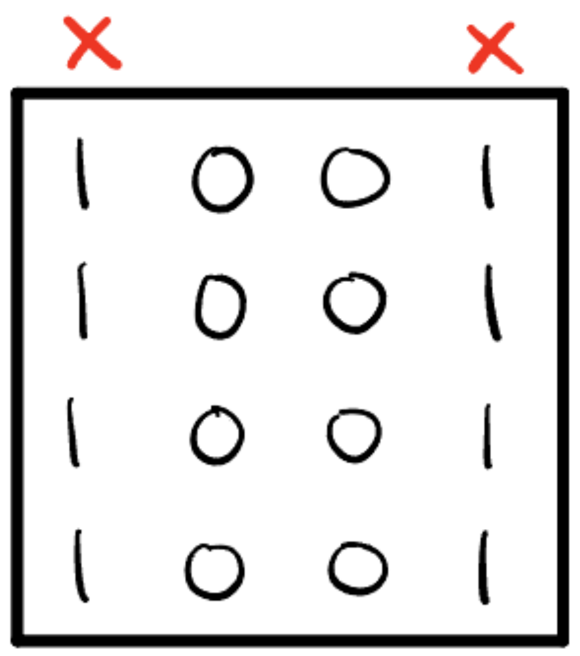

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↳ Total search problems s.t.
verification is by shallow decision-trees.

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↳ Can always rephrase as a false-clause search problem.
Given unsat CNF $C_1 \wedge C_2 \wedge C_3 \wedge \cdots \wedge C_m$
input x
find false clause i.e. $C(x) = 0$

We say these C_is are witnessing for R.

Lower Bounds for t-PPP

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Useful Property

t matchings cannot witness a $(t+1)$-collision

Lower Bound

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t-Collision-freedom special case

$$\tilde{E}[\mathcal{F}] = \sum_{C \in \mathcal{F}} \tilde{E}[C] \leq (t-1)$$ for all t-witnessing families $\mathcal{F}$

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t-witnessing $\mathcal{F}$ family of conjunctions is said to be t-witnessing if

$\forall\, S\subseteq\mathcal{F}$ with $|S|=t$ $\begin{cases} \text{some } C\in S \text{ is witnessing} \\ DT(R\restriction\rho)\le \mathrm{polylog}(n),\ \rho=\bigcup_{C\in S}\rho(C) \end{cases}$

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Matching Pseudodistribution corresponds to a t-Collision-free Pseudoexpectation for $(t+1)$-PIGEON_N^M

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Proof Sketch $t = 2$

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T_4
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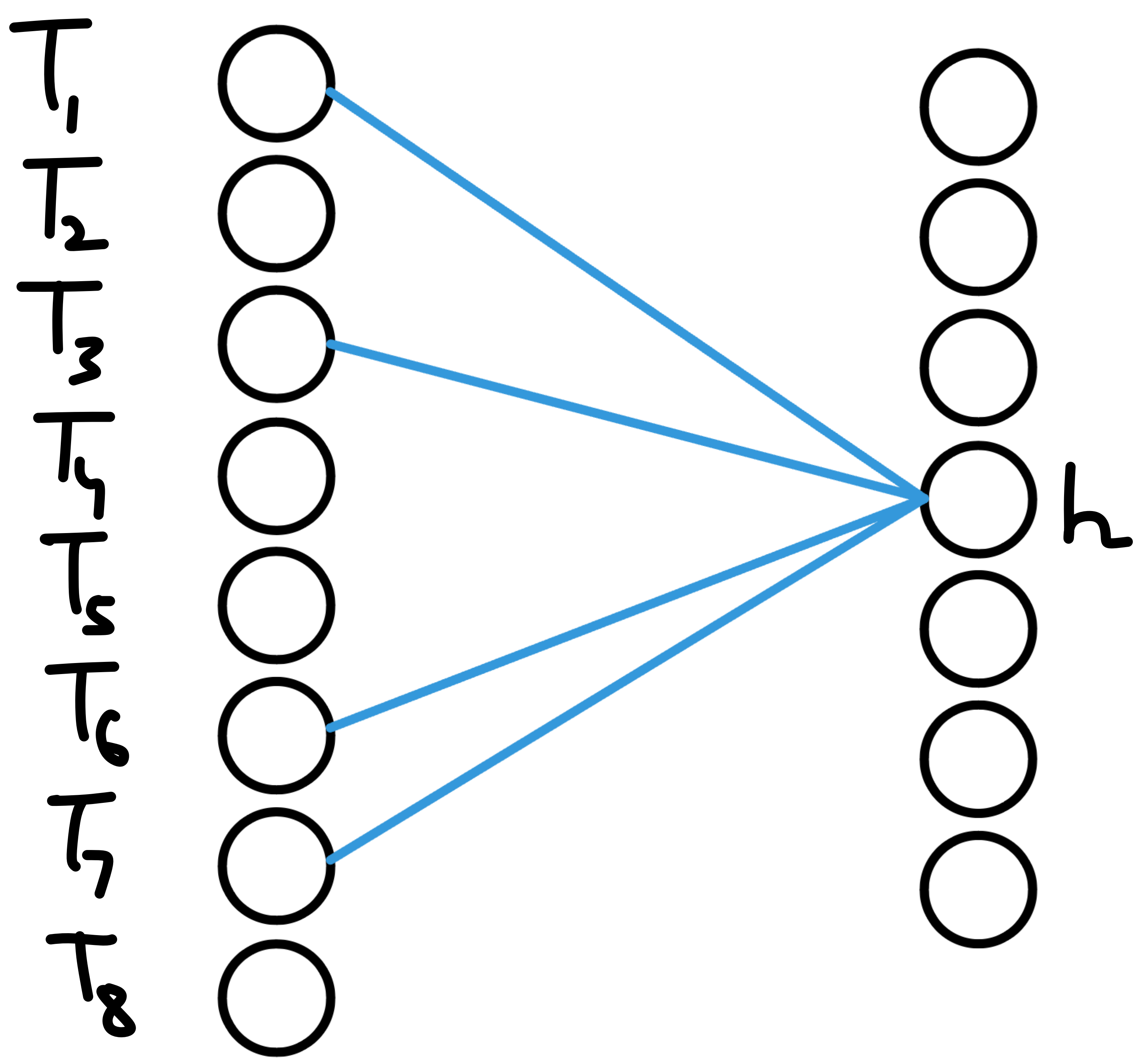

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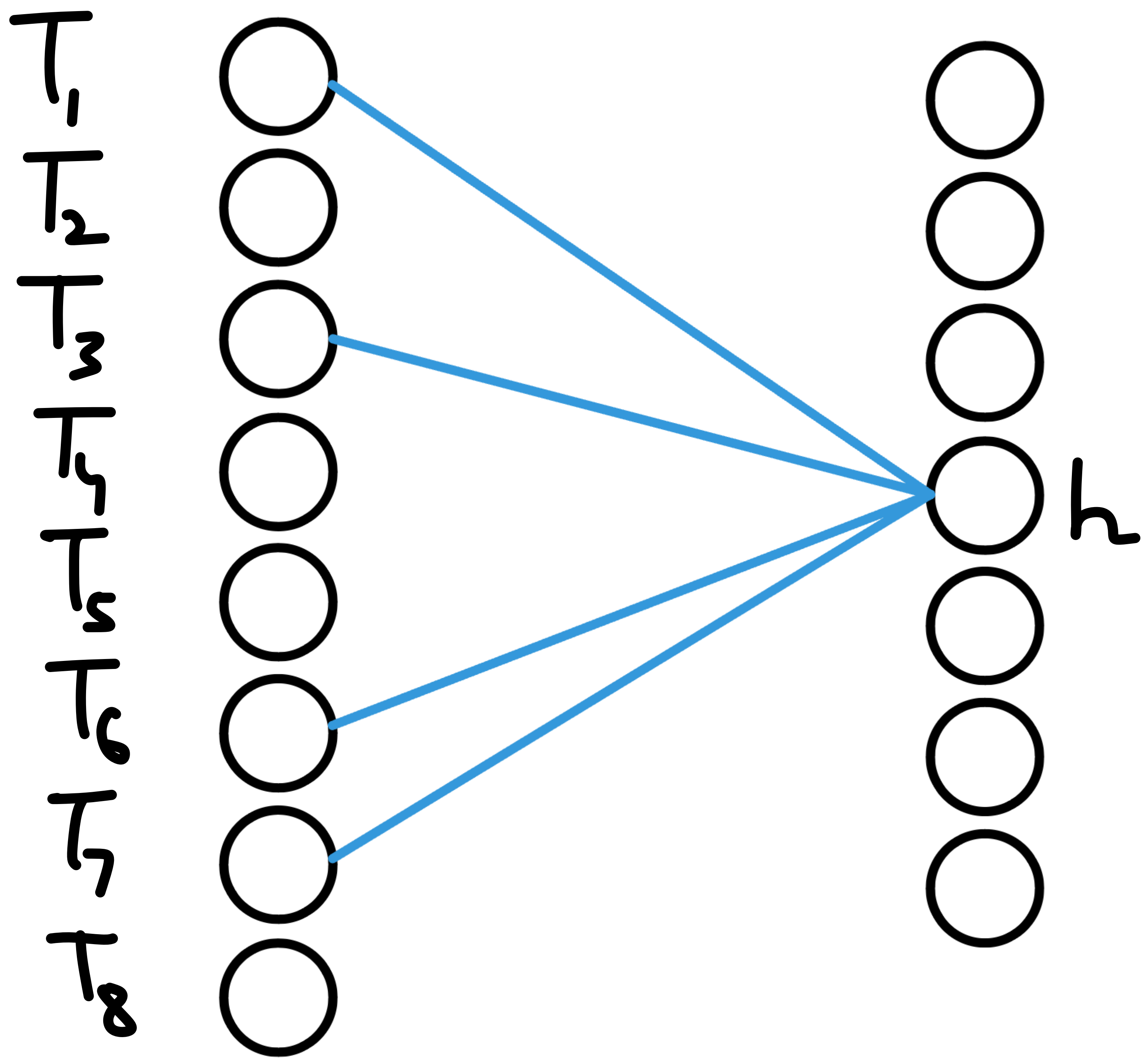

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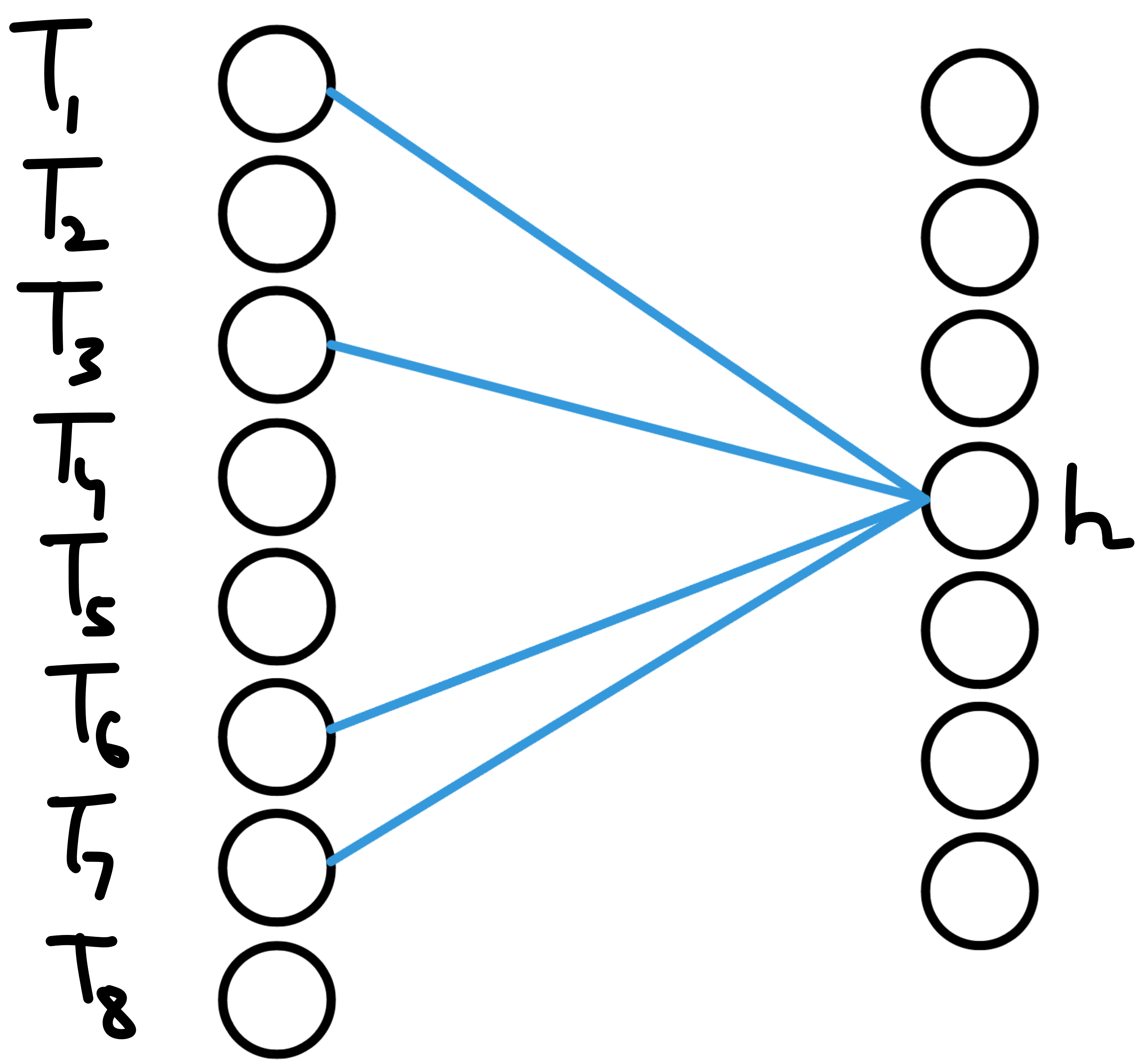

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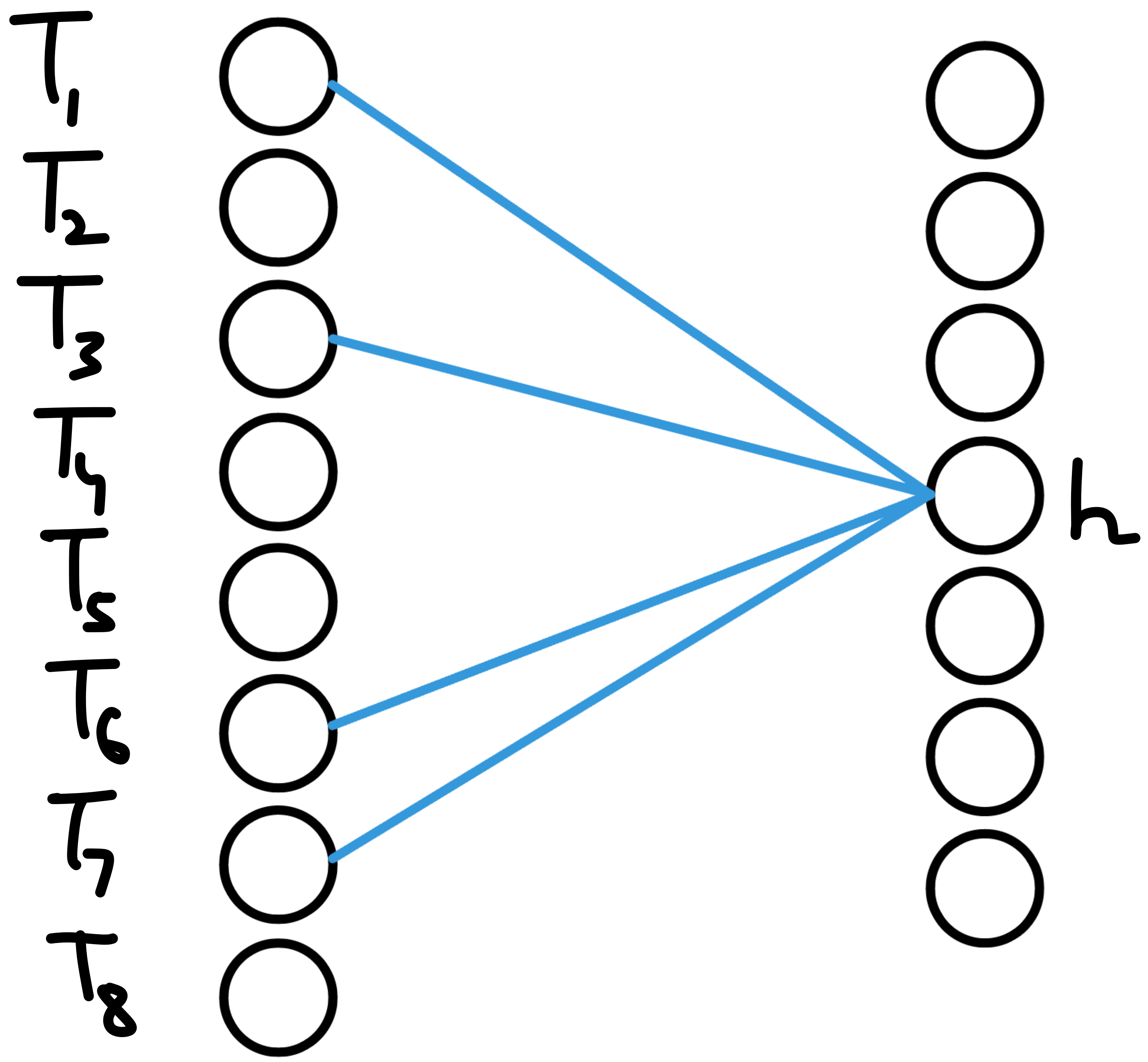

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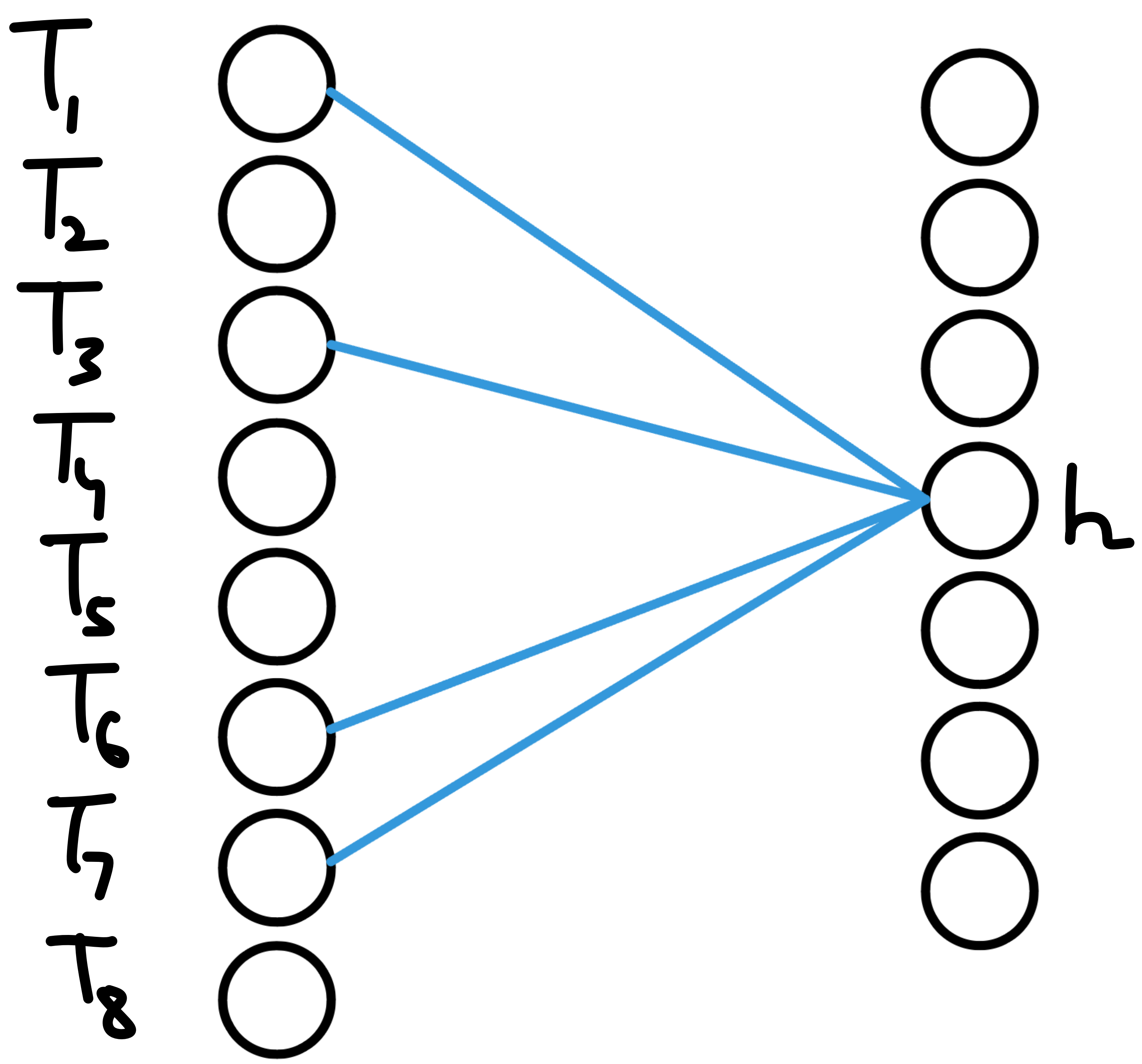

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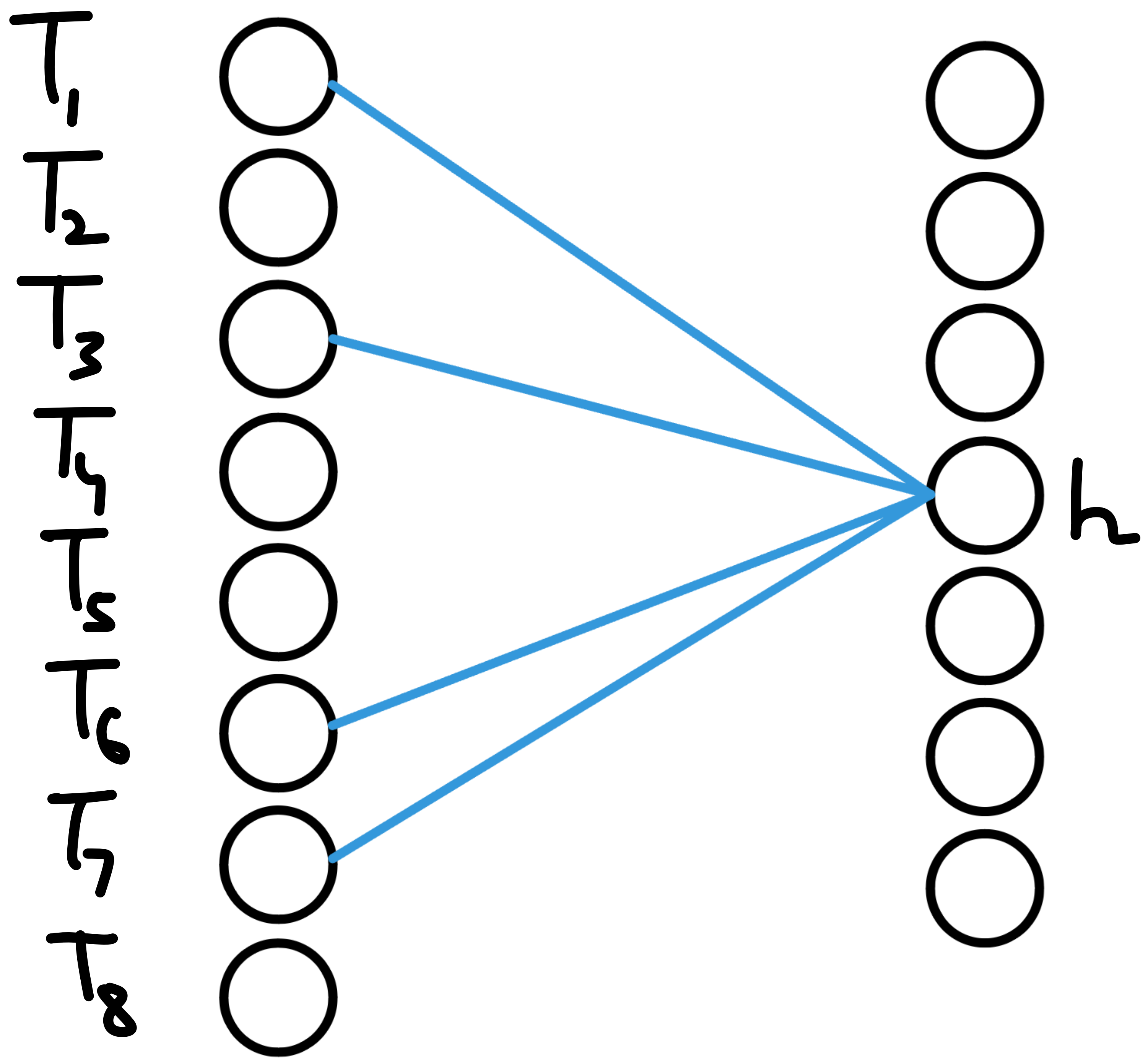

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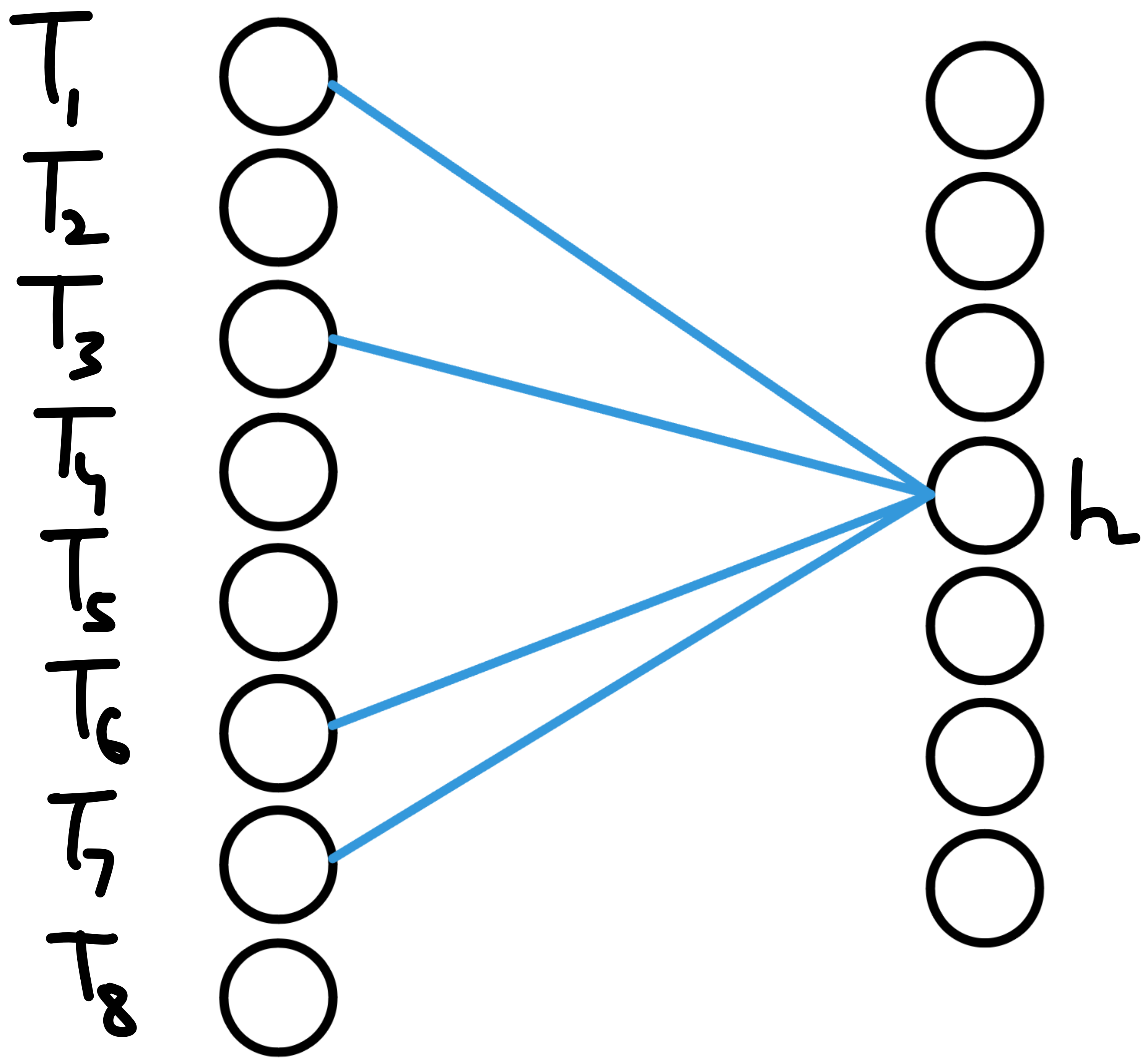

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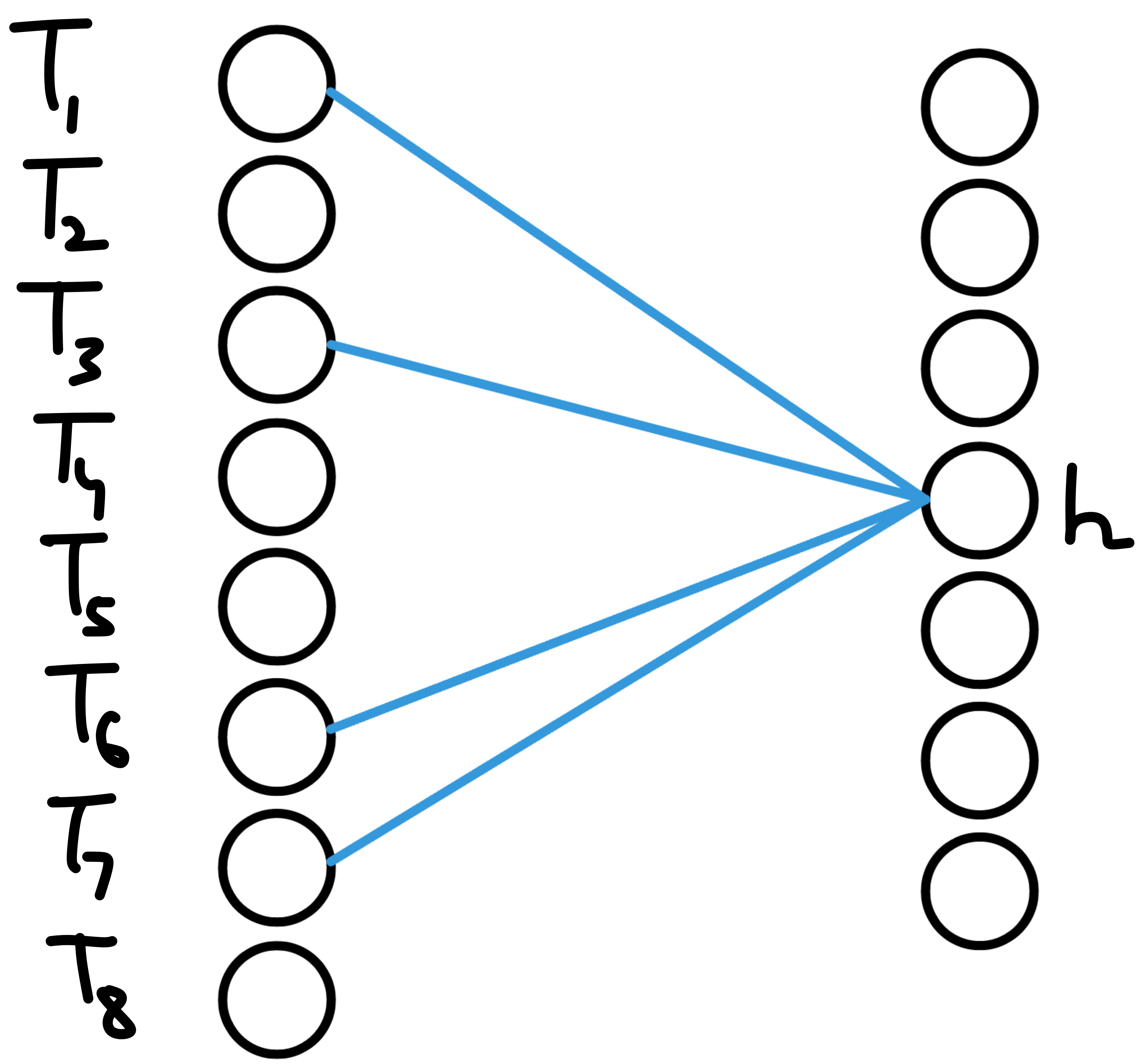

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!

Consequences dt

→ RAMSEY & SAP

Consequences ^{dt}

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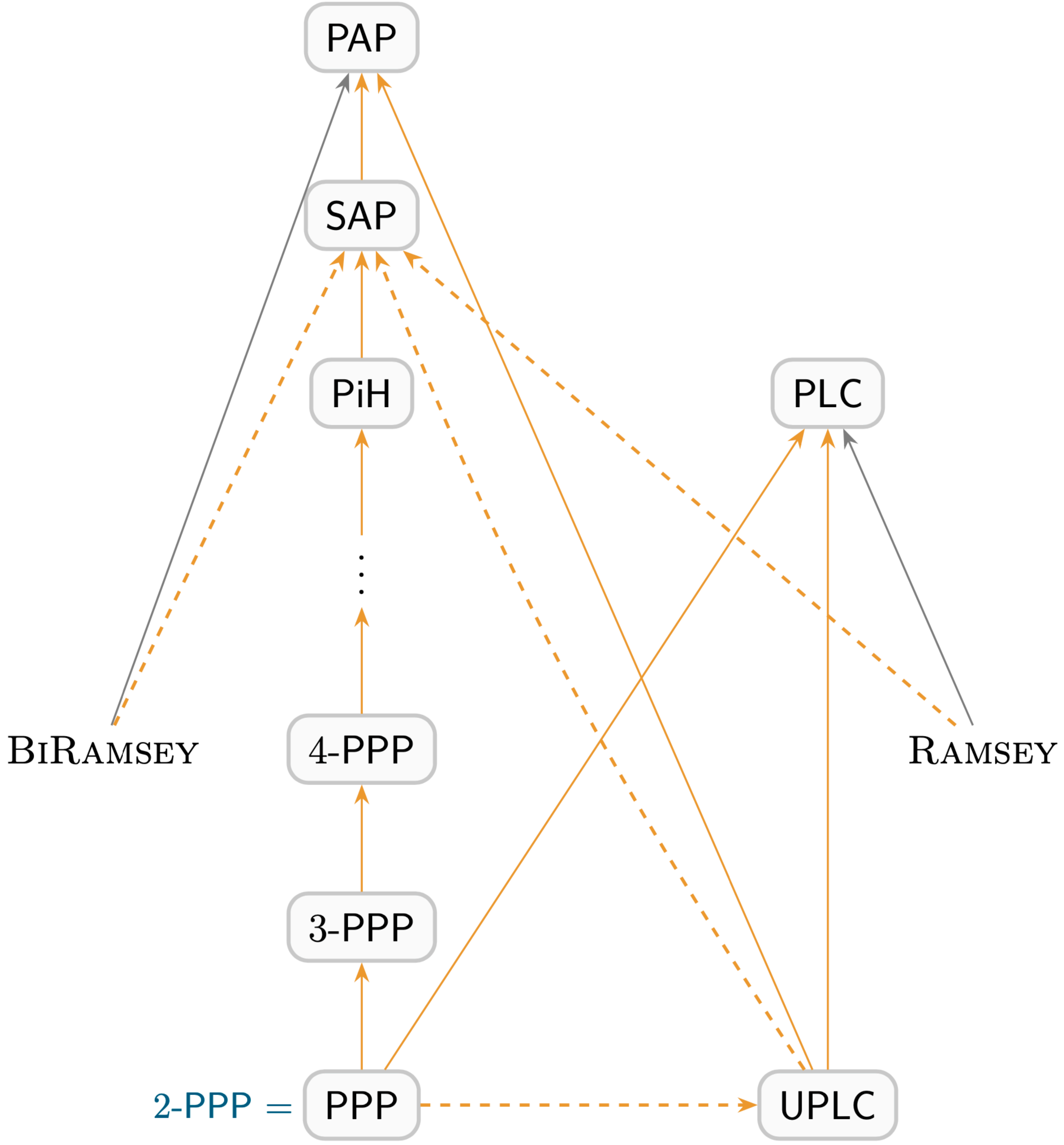

Open Problems

1. RAMSEY $\not\leq$ BIRAMSEY

2. PLC $\not\leq$ PAP (implies 1)

3. t-PIGEON$_N^{2tN}$ vs t-PIGEON$_N^{N^2}$

4. MCRH vs CRH

Thanks for listening!

Au revoir