Communication Complexity of Collision

Mika Göös (EPFL), Siddhartha Jain (EPFL → UT Austin)

Main result

CC lower bound for NATURAL two-party Collision

Collision

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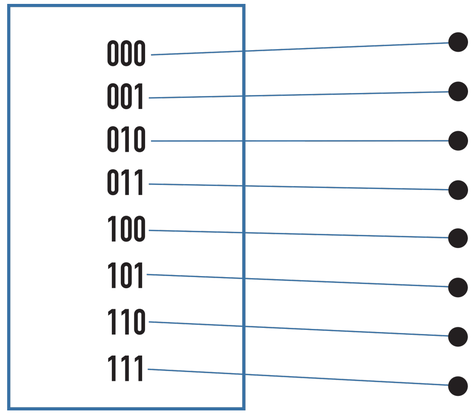
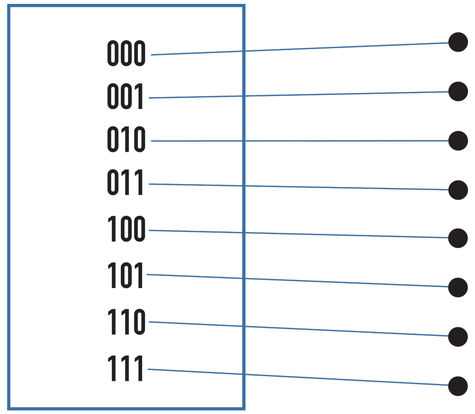
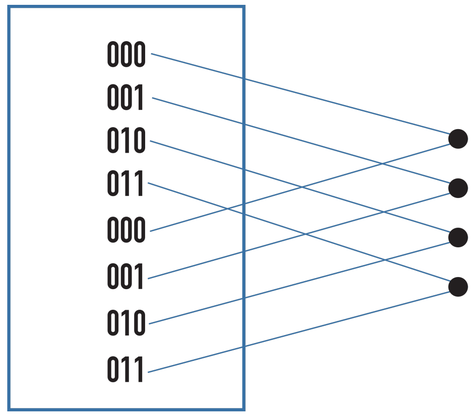


Collision

Given a list of N numbers in $[N]$, say z. $COL_N(z)$ is a partial function only defined when z is 1-1 or 2-1. Well studied, motivated by cryptanalysis.





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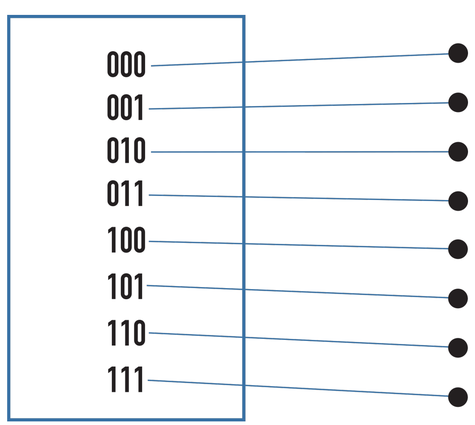

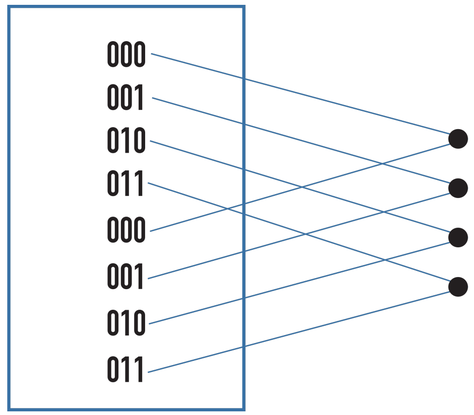

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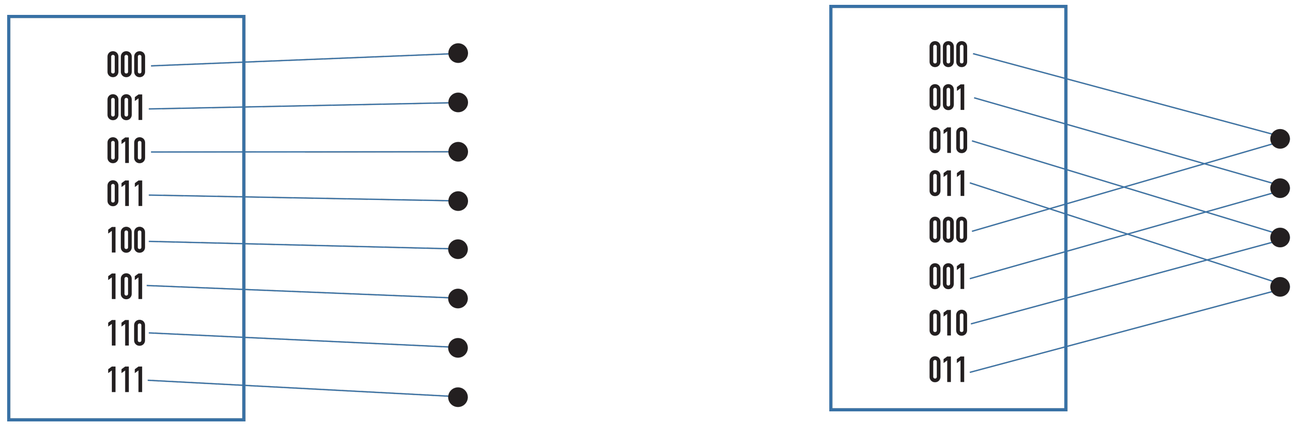

- Randomised query complexity = $\theta(\sqrt{n})$ (folklore)
- Quantum query complexity = $\theta(n^{1/3})$ [Aar02, AS04]

Bipartite Collision

Image taken from https://github.com/greeenway/alicebobandeve

Bipartite Collision

Now we split the binary encoding of each number in z to get two lists a, b.
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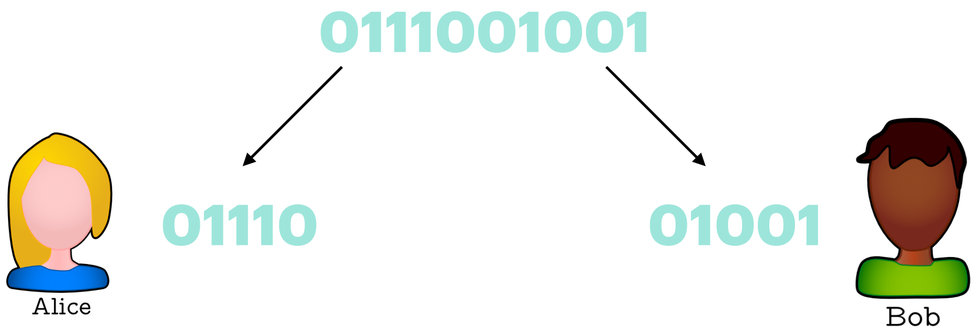

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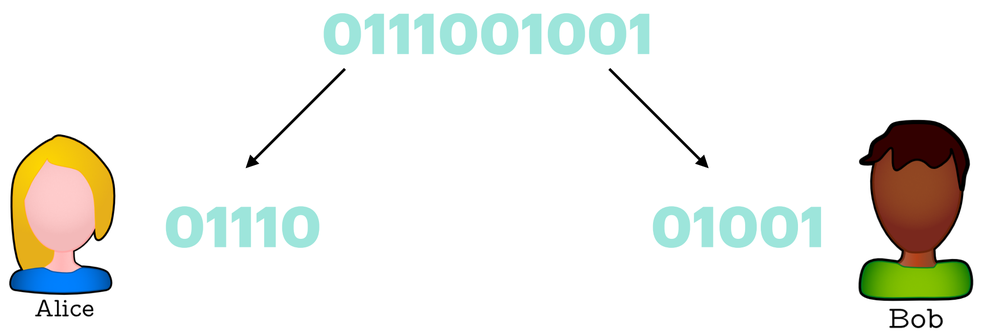

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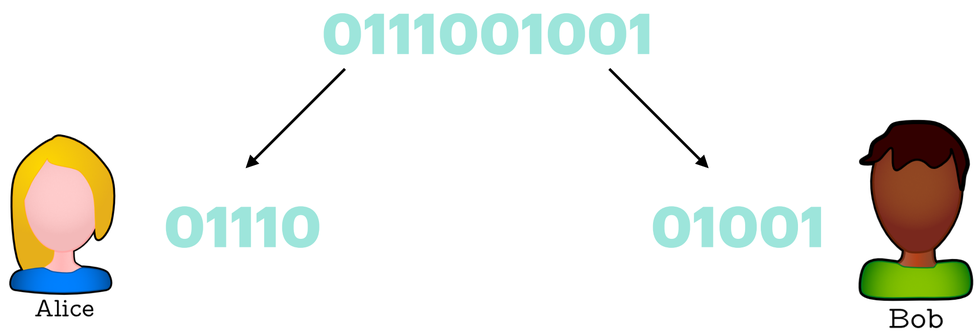

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Main theorem. $BICOL_N$ has randomised (and even quantum) communication complexity $\Omega(N^{1/12})$.

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Communication and lifting

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Technical barrier. Popular method to show communication lower bounds is *lifting.* Given f for which we know a query lower bound, we wish to compose with a small “gadget" g to create a two-party problem.

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$g = \mathrm{XOR}$

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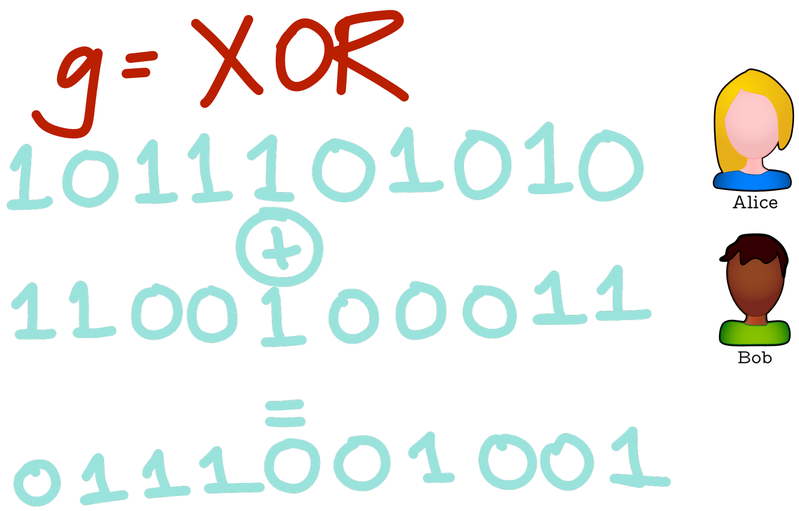

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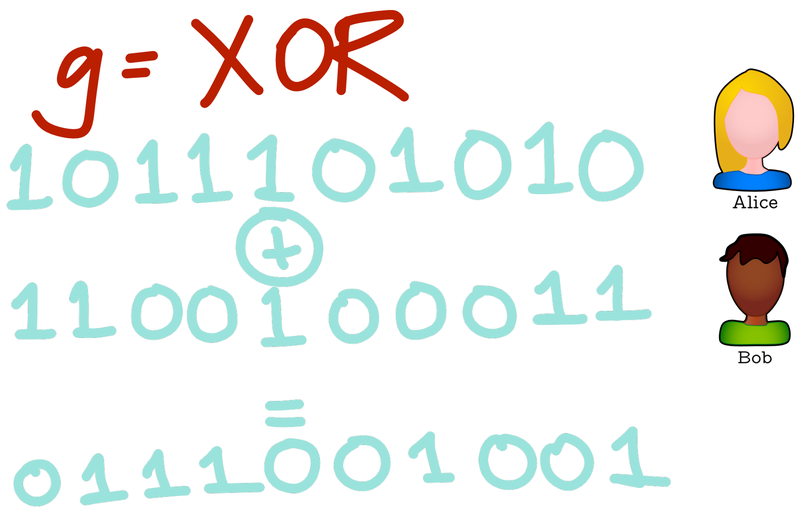
v. s. natural problem $BICOL_N$



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Alice

Bob

a_1	b_1
a_2	b_2
…	…
a_i	b_i
…	…
a_j	b_j
…	…
a_m	b_m

$$a_i + b_i = a_j + b_j$$

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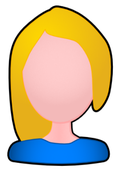
Alice


Bob

a_1	b_1	$[a_1 + z]$	$[b_1 + z]$
a_2	b_2	$[a_2 + z]$	$[b_2 + z]$
…	…	…	…
a_i	b_i	$[a_i + z]$	$[b_i + z]$
…	…	…	…
a_j	b_j	$[a_j + z]$	$[b_j + z]$
…	…	…	…
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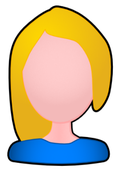

$$
\begin{array}{cc}
a_1 & b_1 \\
a_2 & b_2 \\
\dots & \dots \\
a_i & b_i \\
\dots & \dots \\
a_j & b_j \\
\dots & \dots \\
a_m & b_m
\end{array}
\longrightarrow
\begin{array}{cc}
[a_1 + z] & [b_1 + z] \\
[a_2 + z] & [b_2 + z] \\
\dots & \dots \\
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\dots & \dots \\
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[a_m + z] & [b_m + z]
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$a_i + z_0 = a_j + z_1,$
$b_i + z_0 = b_j + z_1$

$\Longrightarrow$

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z_0, z_1 unique pair!

$a_i + b_i = a_j + b_j$

$(a_i + z_0, b_i + z_0)$
$= (a_j + z_1, b_j + z_1)$

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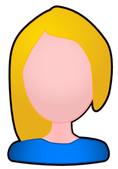

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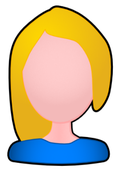

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CLAIM

Only using the "regularity" of XOR

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XOR is Regular

$(x, y) \mapsto (x, y)$
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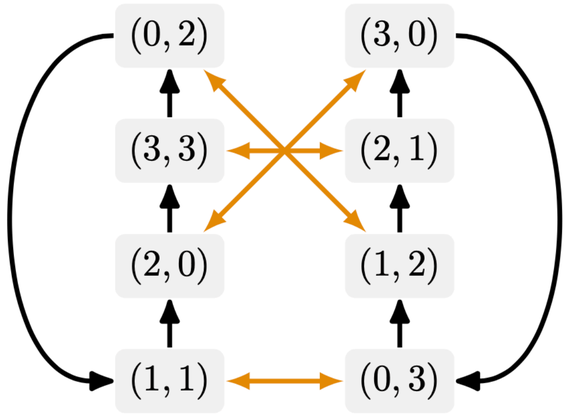
	0	**1**	**2**	**3**
0	0	0	1	1
1	0	1	1	0
2	1	1	0	0
3	1	0	0	1

(a)



(b)

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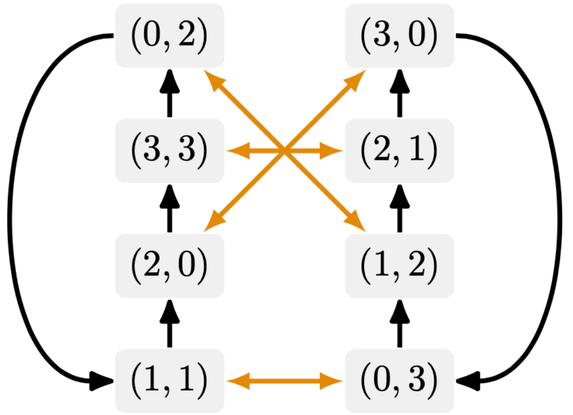

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Generators on $VER^{-1}(1)$

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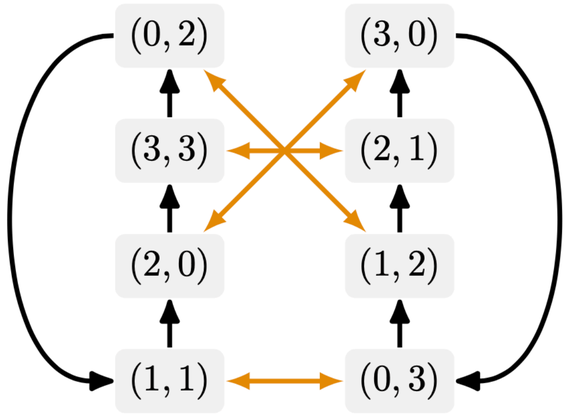

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$VER : \mathbb{Z}_4 \times \mathbb{Z}_4 \mapsto \{0,1\}$

Generators on $VER^{-1}(1)$

(black) $(x, y) \mapsto (x + 1, y - 1)$

(orange) $(x, y) \mapsto (1 - x, -y)$

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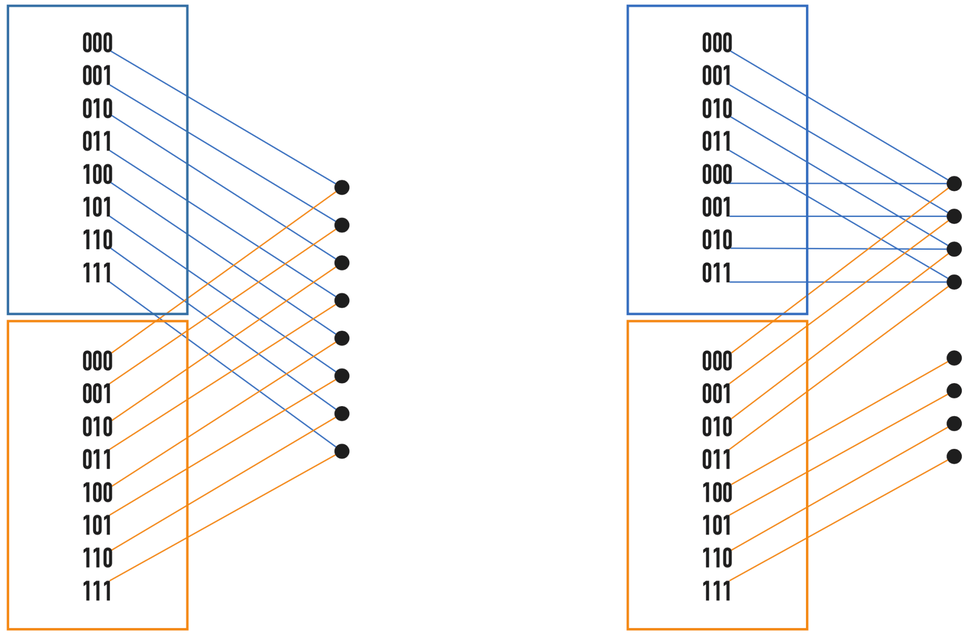

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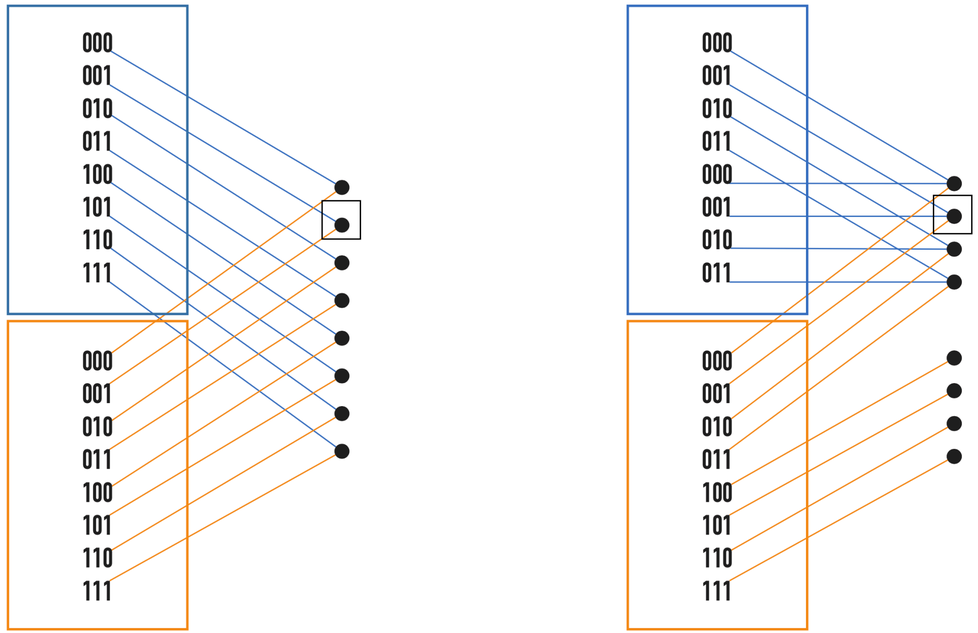

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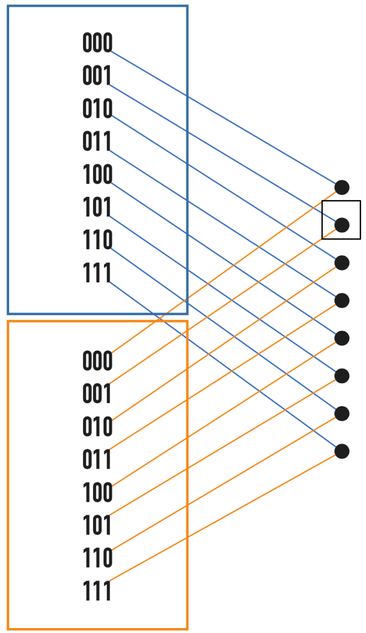

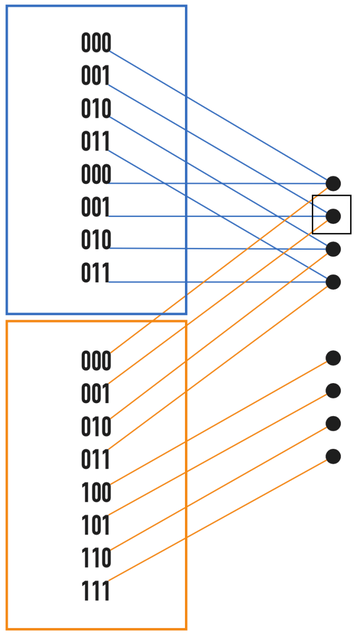

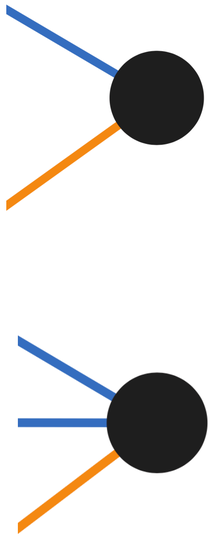

Thanks for listening!

Au revoir