Separations in Proof Complexity and TFNP

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Understanding The Title

TFNP := Total Function NP

Polytime $R(x,y)$

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Input x

Output $y : R(x,y) = 1$ & $|y| \leq |x|^{O(1)}$

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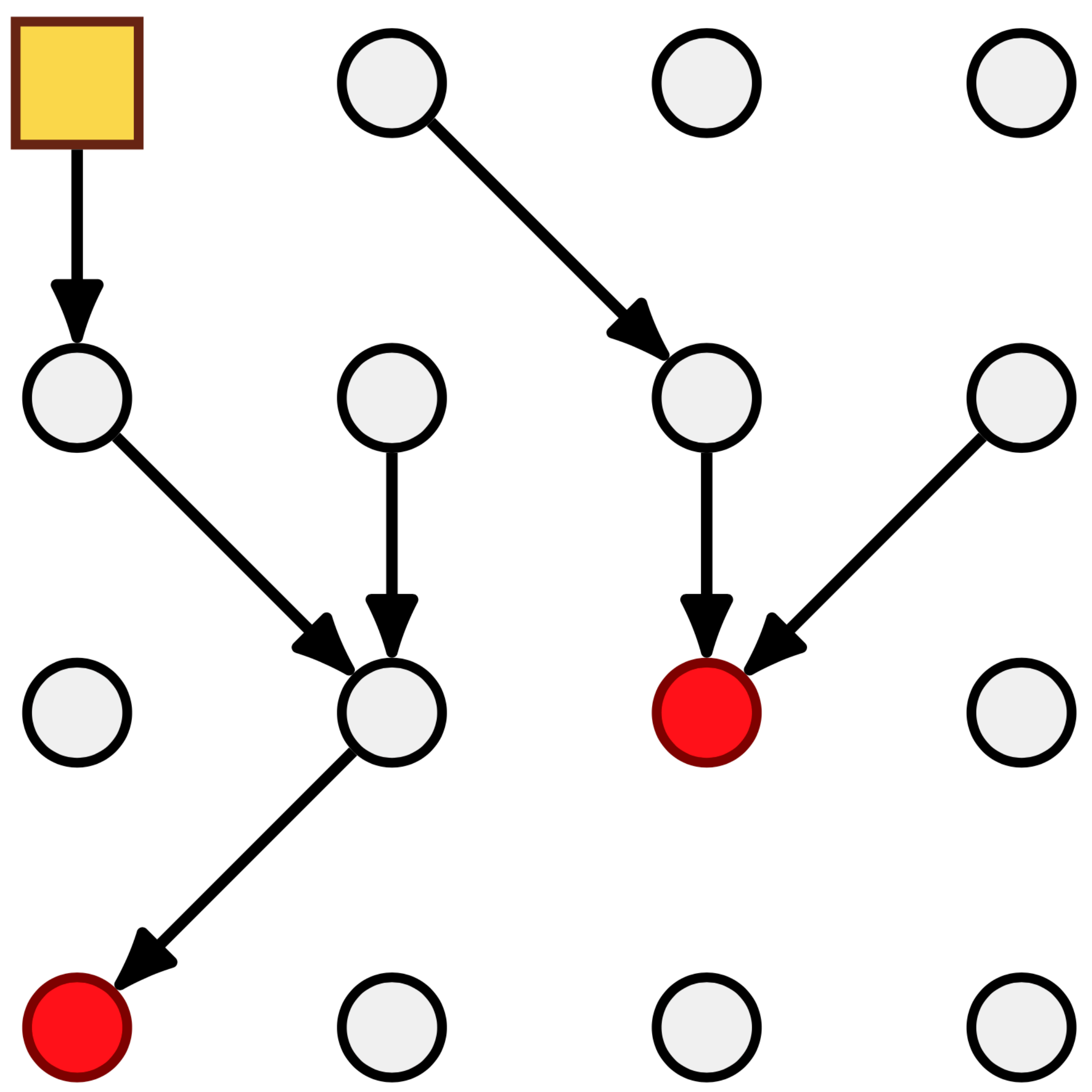
Polytime $R(x,y)$

Input x

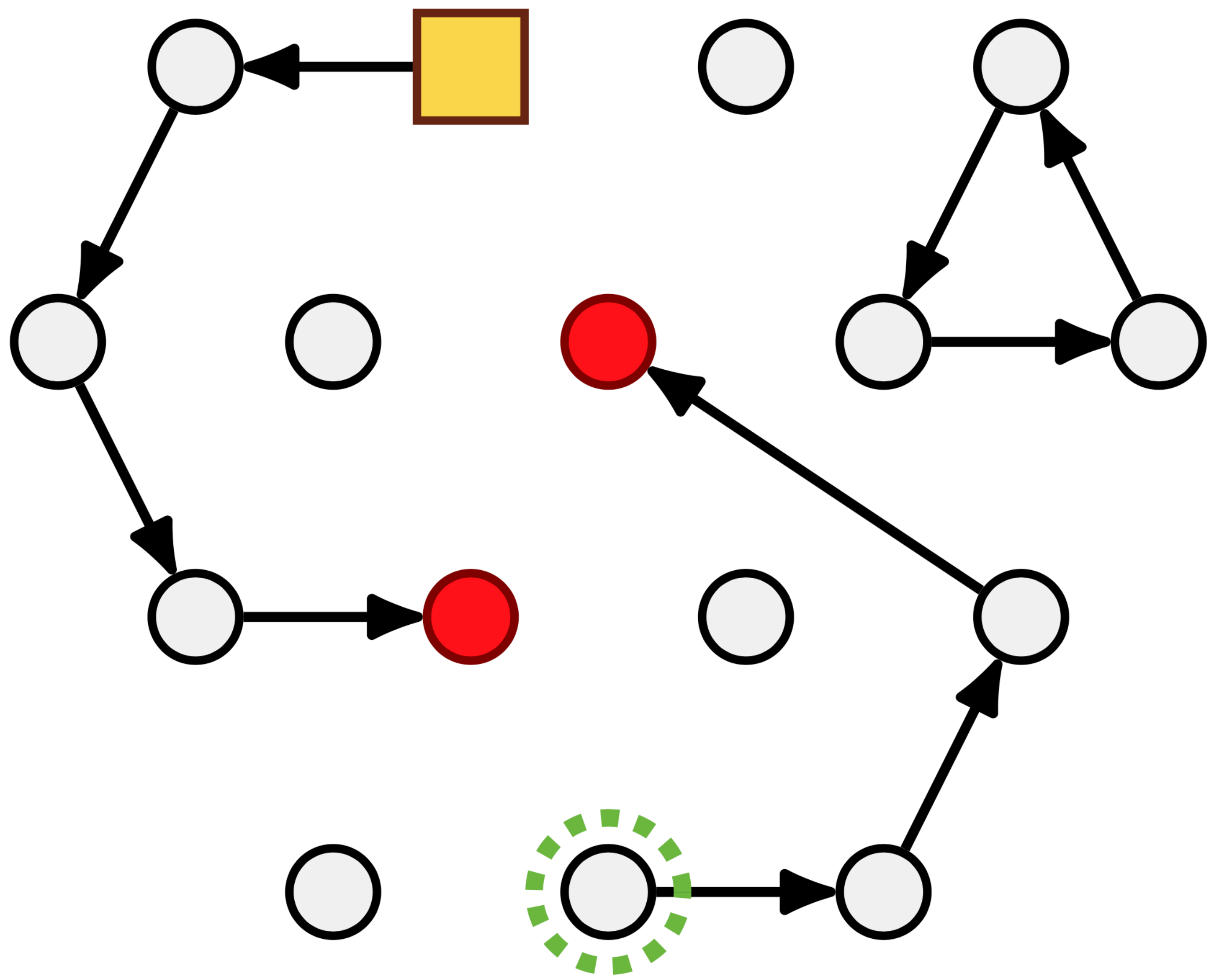
Output $y : R(x,y) = 1$ & $|y| \leq |x|^{O(1)}$

Promise R is total: $\forall x\, \exists y\; R(x,y) = 1$

Two Problems

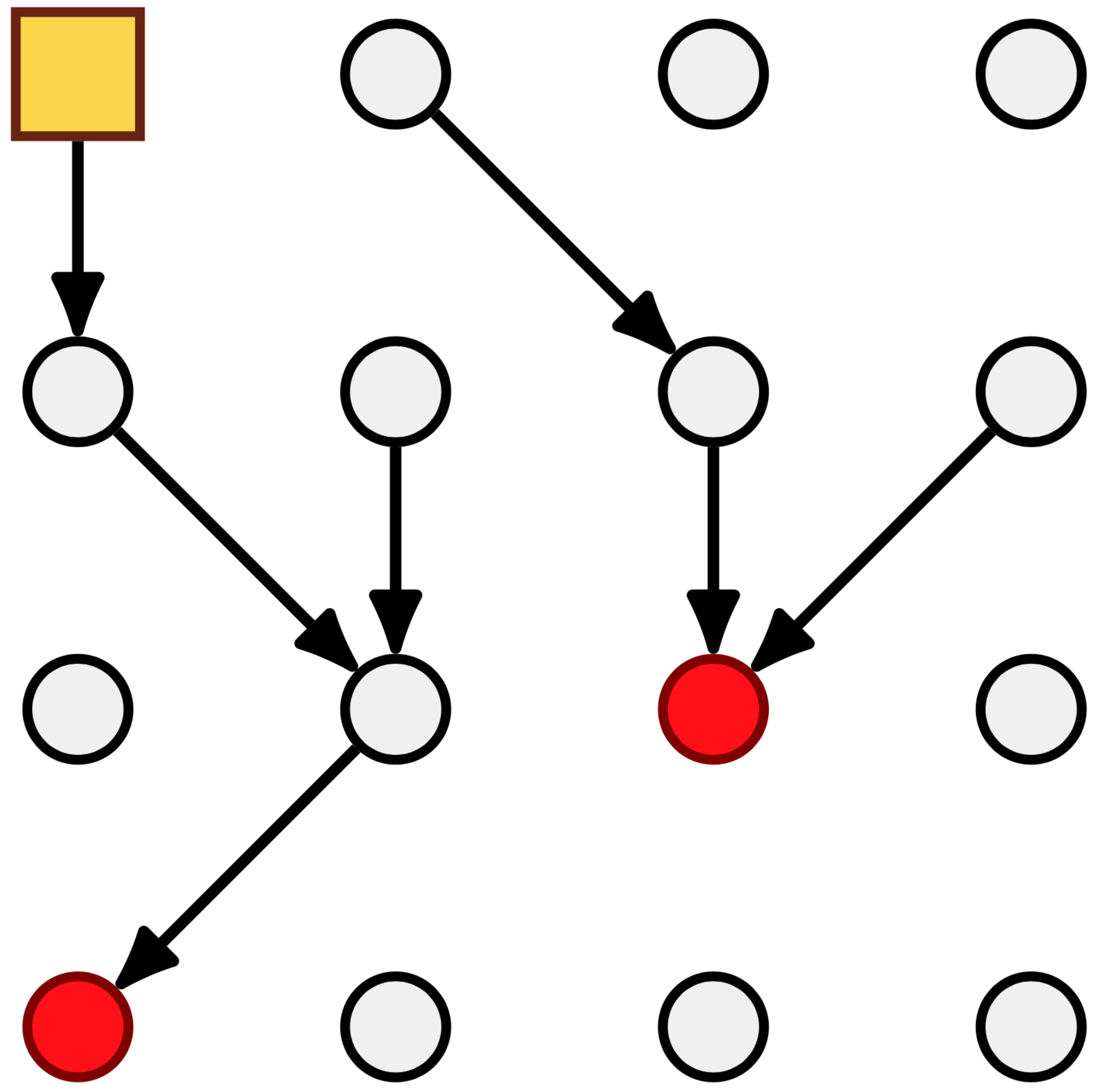


Sink-of-DAG (SoD)

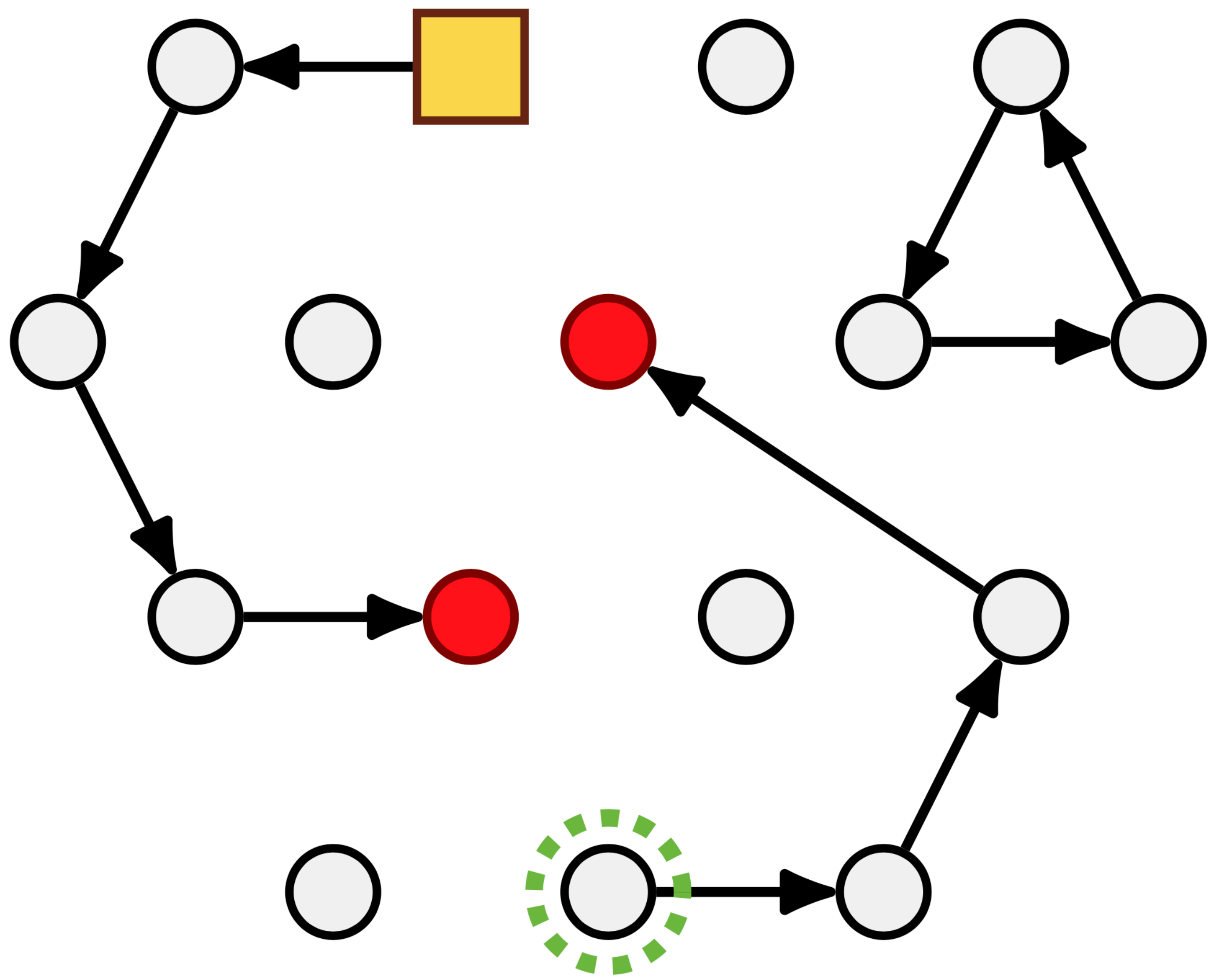


Sink-of-Line (SoL)

Two (& ½) Problems



Sink-of-DAG (SoD)



Sink-of-Line (SoL)
End-of-Line (EoL)

... And Three Classes

$$\mathrm{PLS} = \{ P : P \leq \mathrm{S \circ D} \}$$

$$\mathrm{PPADS} = \{ P : P \leq \mathrm{S \circ L} \}$$

$$\mathrm{PPAD} = \{ P : P \leq \mathrm{E \circ L} \}$$

… And Three Classes

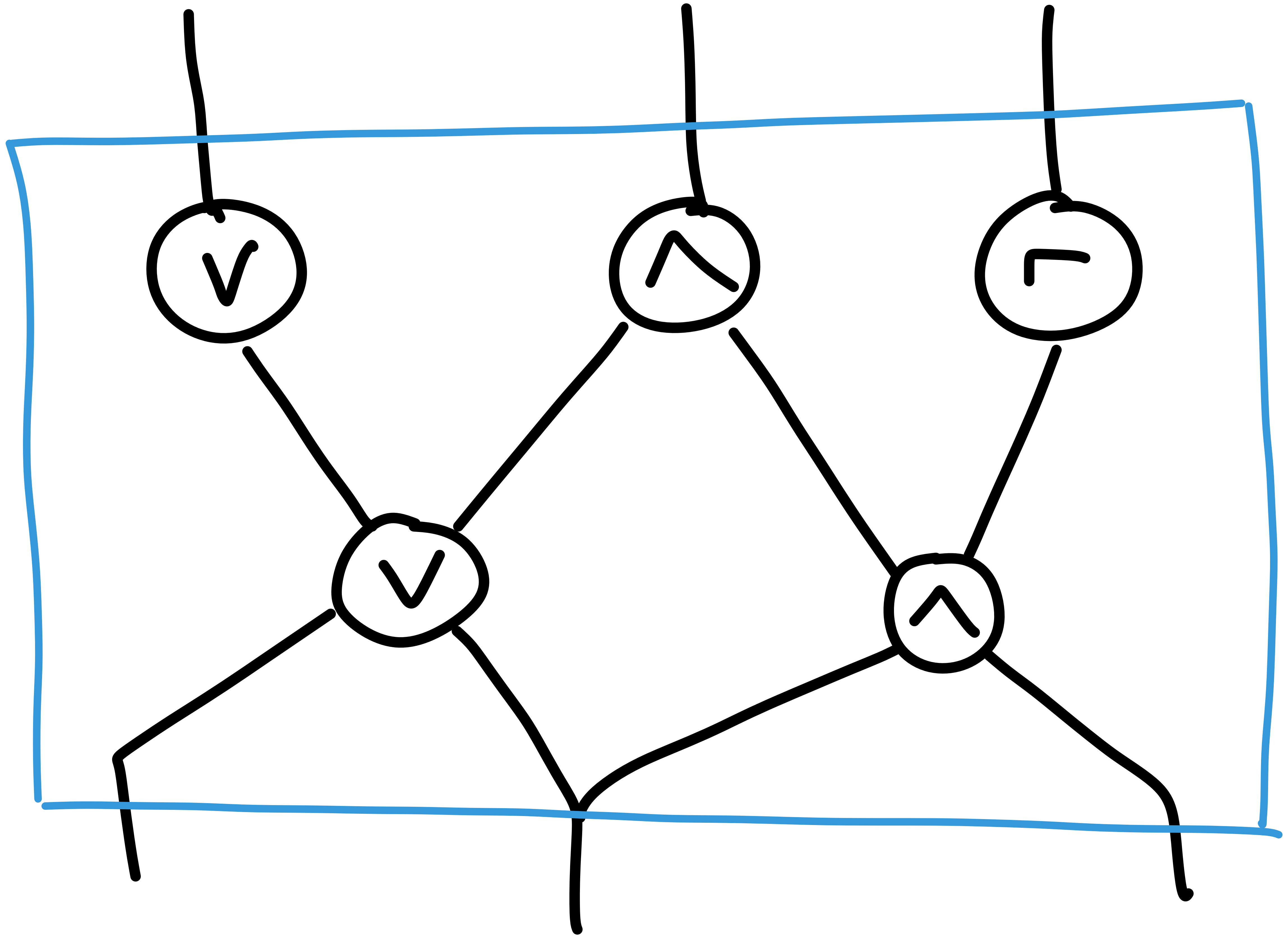
$$\mathrm{PLS} = \{ P : P \leq \mathrm{SoD} \}$$

$$\mathrm{PPADS} = \{ P : P \leq \mathrm{SoL} \}$$

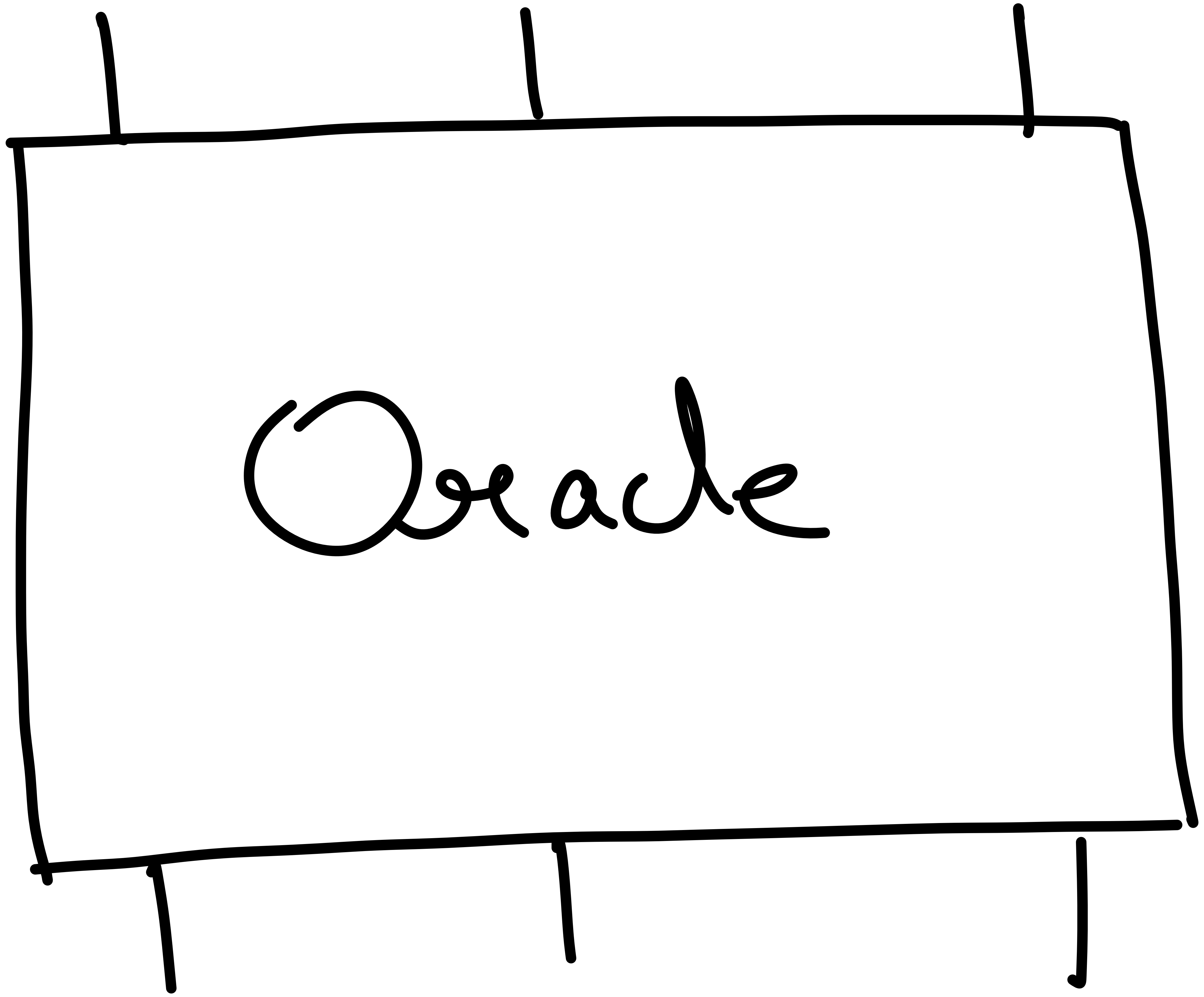
$$\mathrm{PPAD} = \{ P : P \leq \mathrm{EoL} \}$$

$A \leq B$ if $\exists\, R_1, R_2$

$$\begin{tikzcd}
x \arrow[r, "R_1"] \arrow[d, dashed, "\text{Solves } A"'] & x' \arrow[d, "\text{Solve } B"] \\
y & y' \arrow[l, "R_2"]
\end{tikzcd}$$

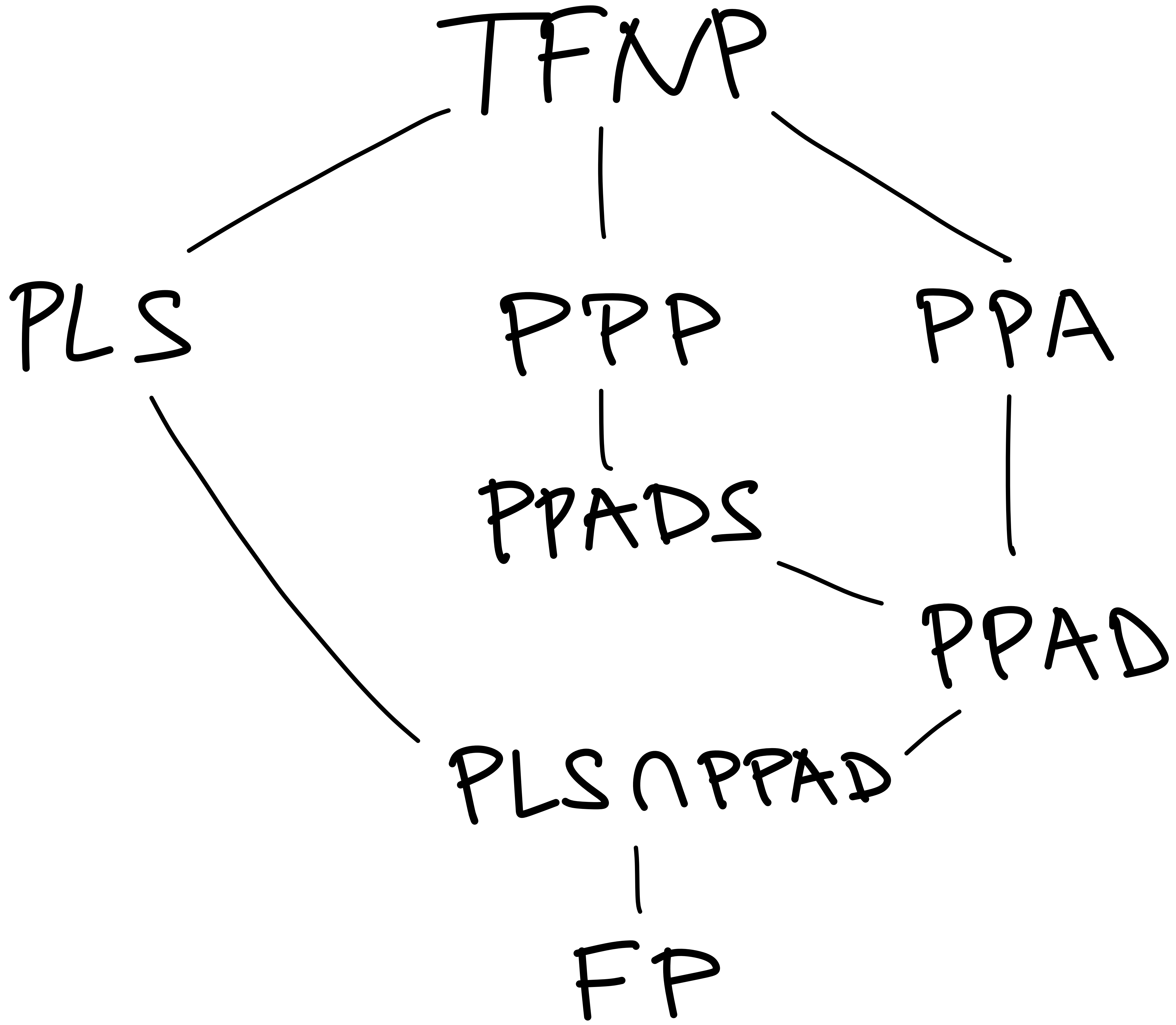


White-box



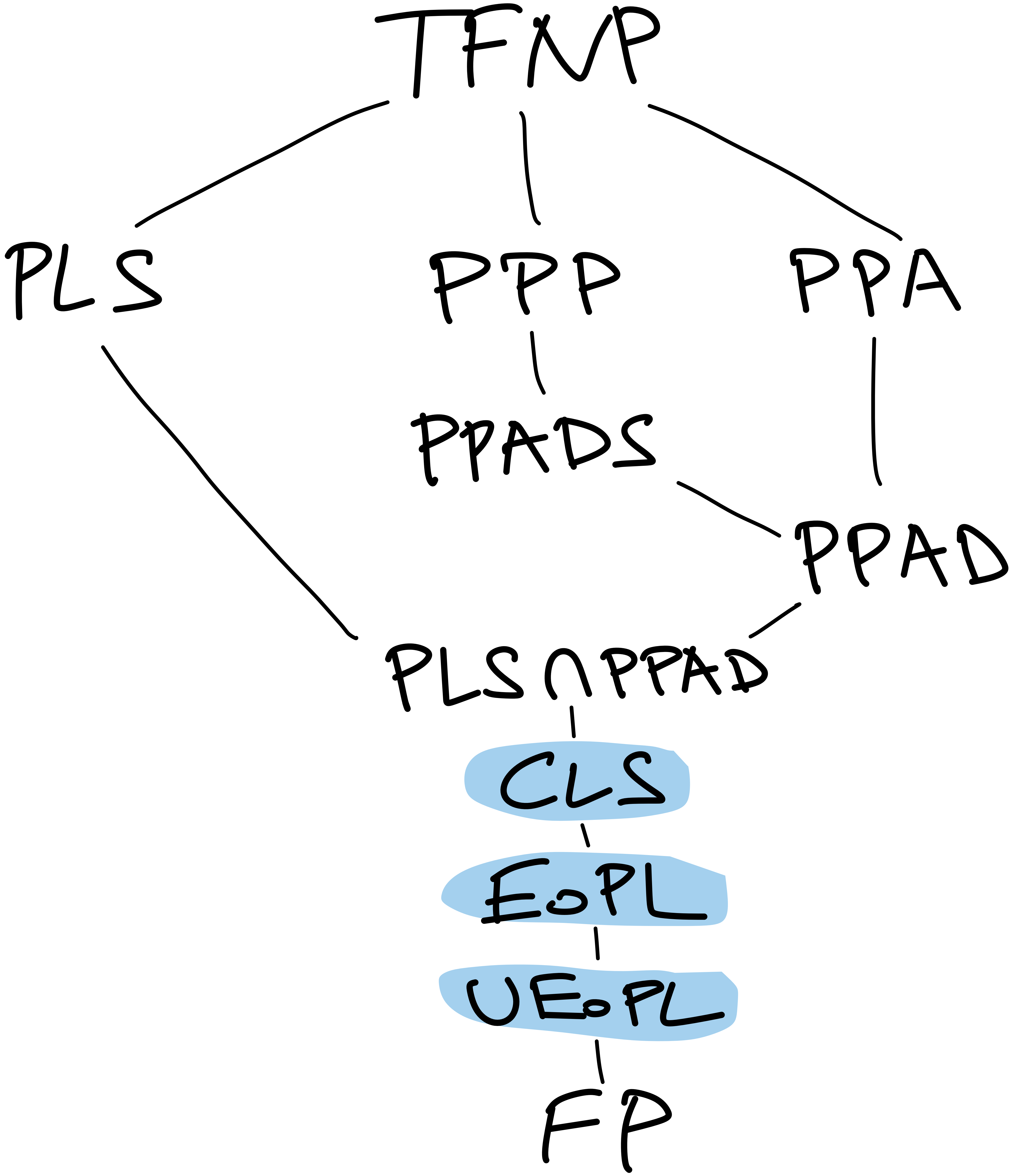
Black-box

Classical hierarchy (90's and 00's)



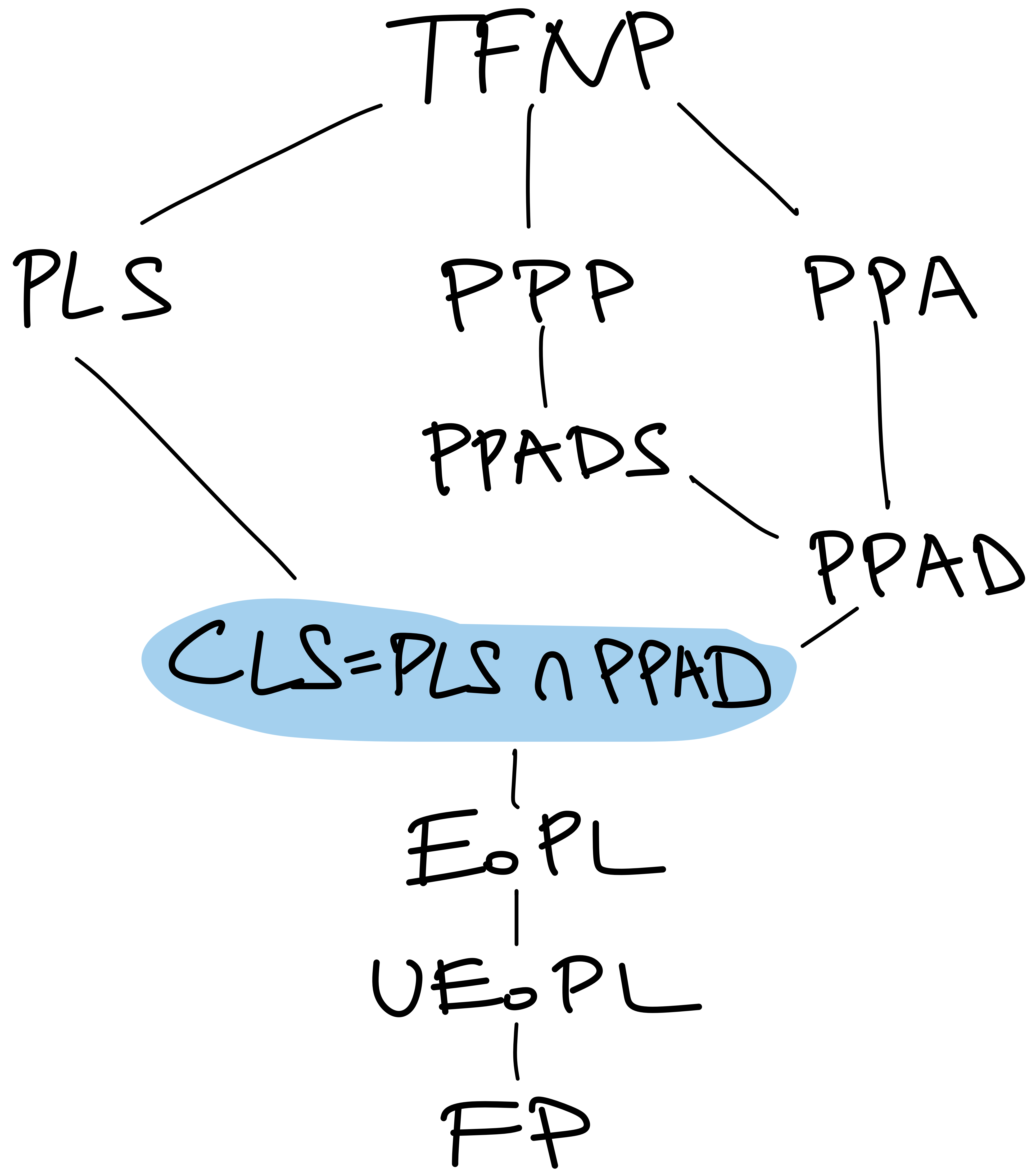
[Pap94]
[JPY88]

New classes (10's)



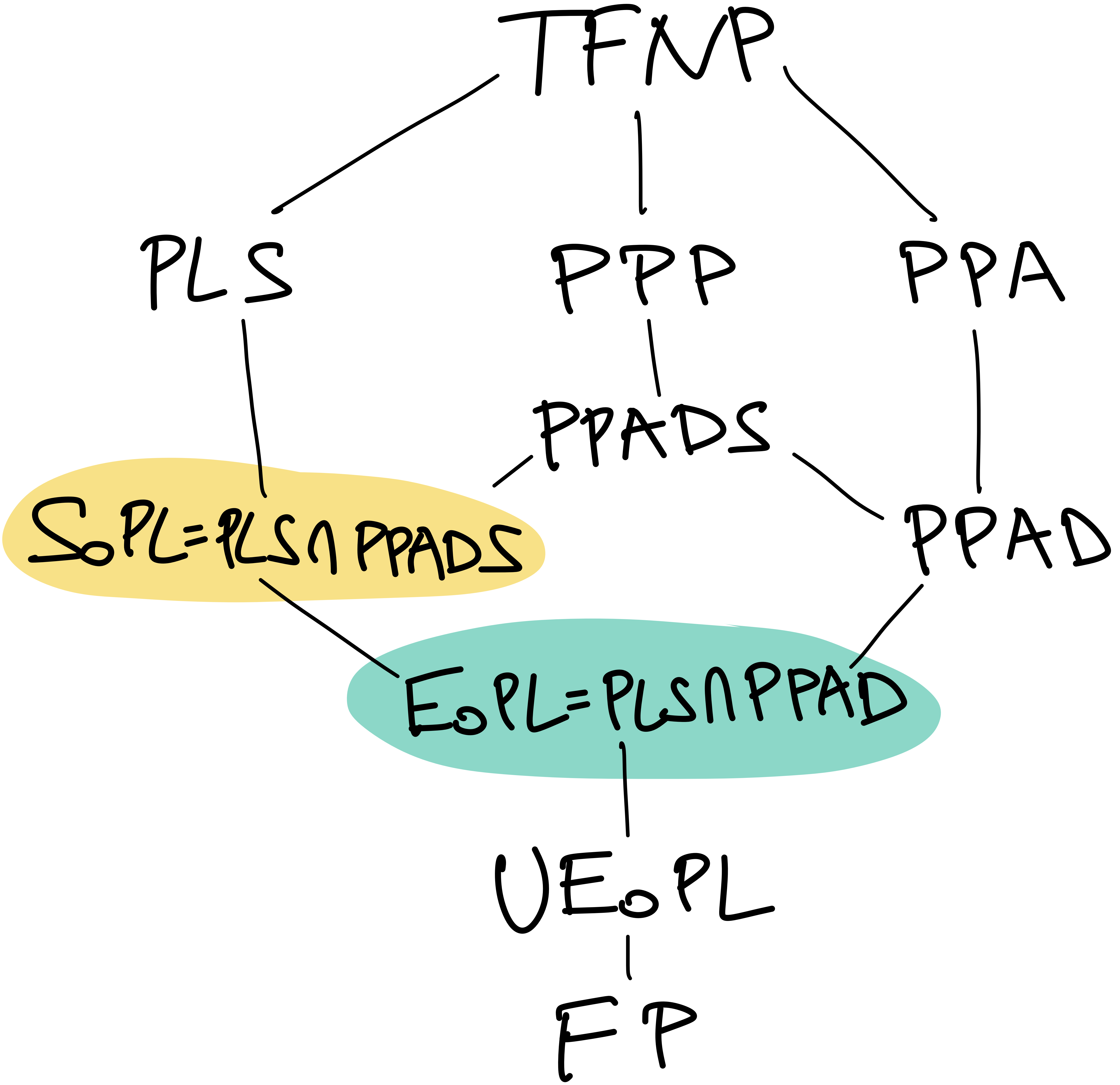
[HY20]
[FGMS20]
[DP11]

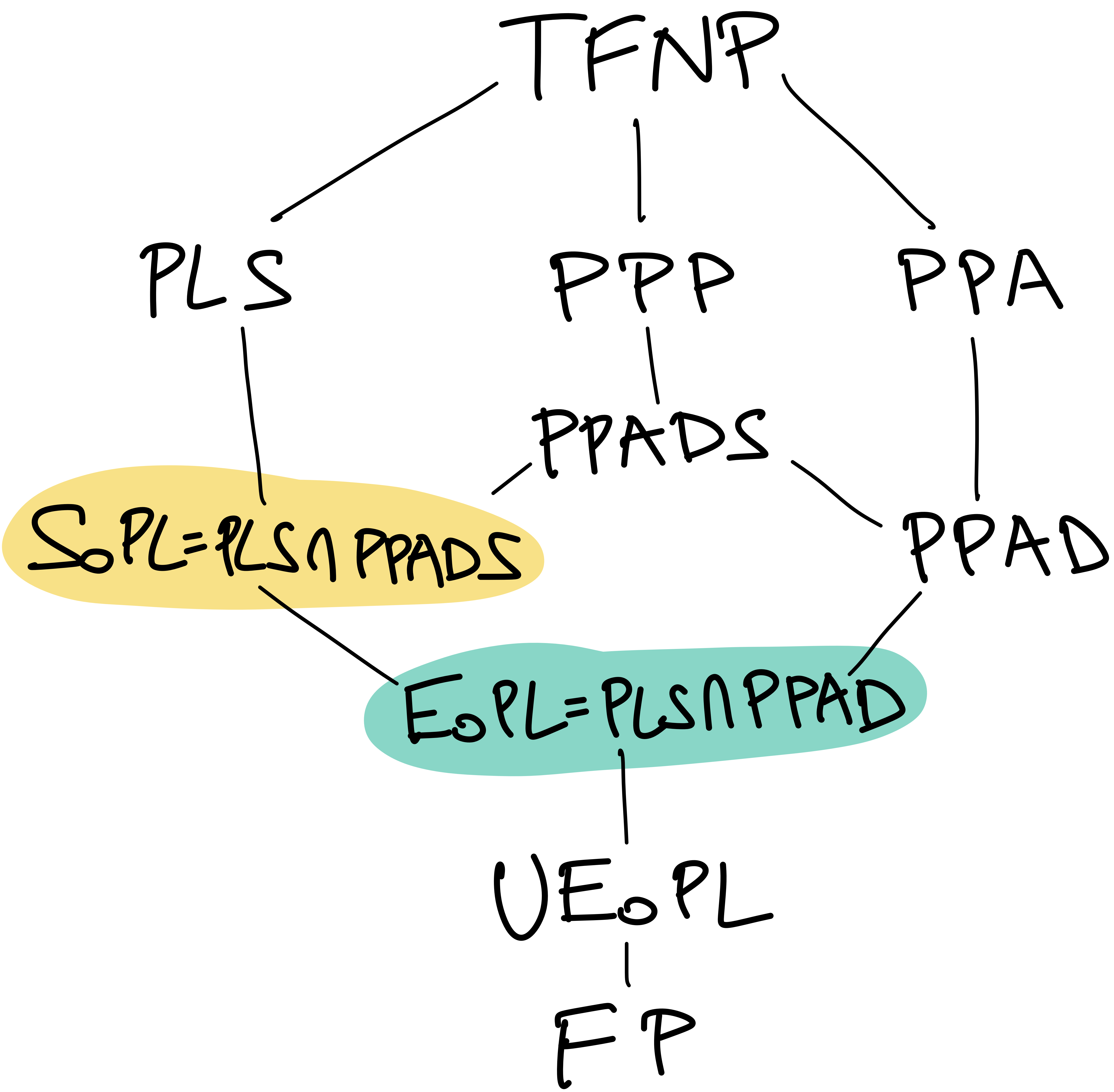
A Breakthrough Collapse (2021)



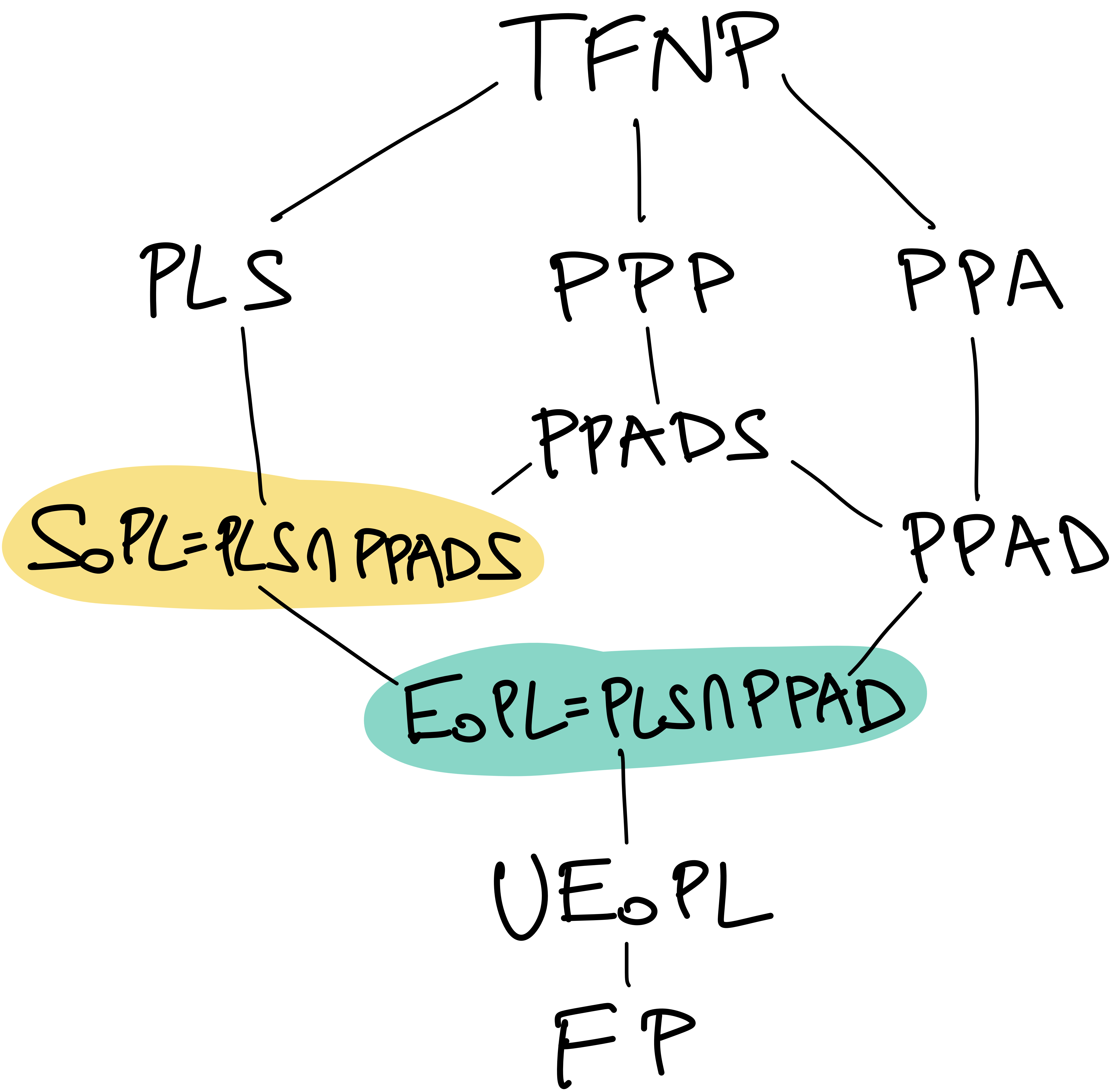
(Best paper!)
[FGHS21]

Further Collapses (2022)





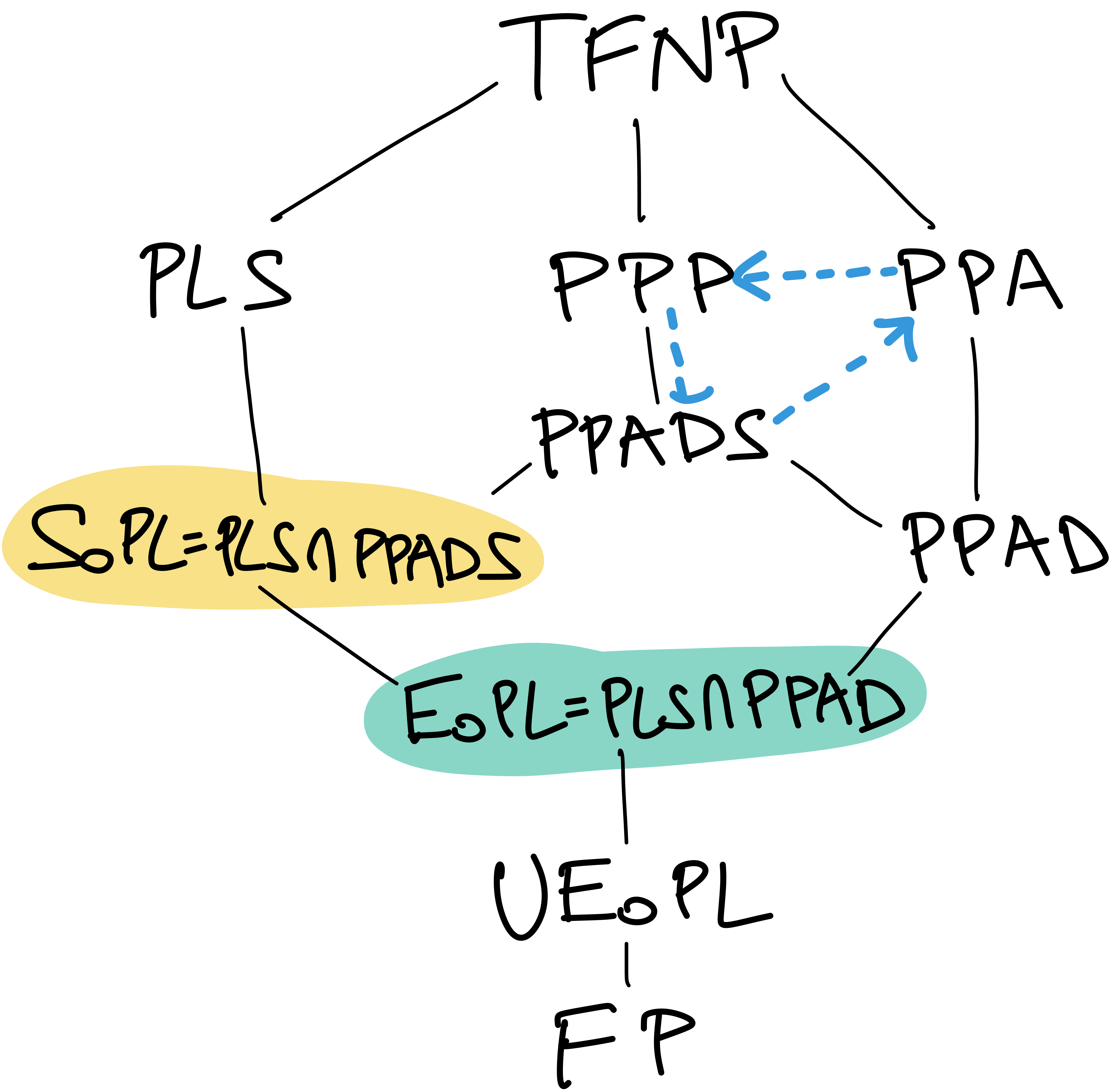
More Collapses?



More Collapses?

White-box sep. $\Rightarrow$ P $\neq$ NP

Black-box sep. possible



More Collapses?

White-box sep. $\Rightarrow P \neq NP$

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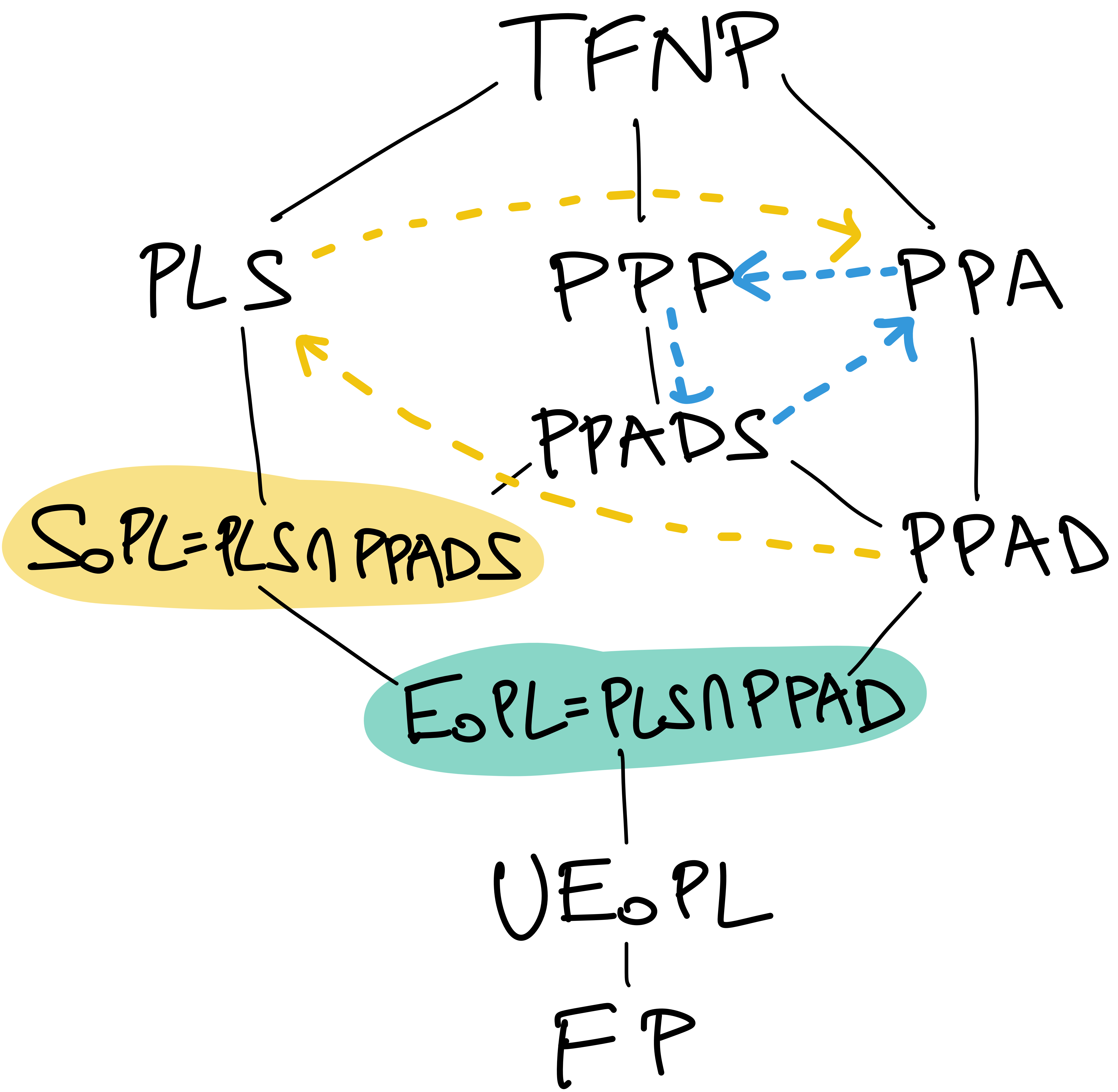
Beame et al. 98'



More Collapses?

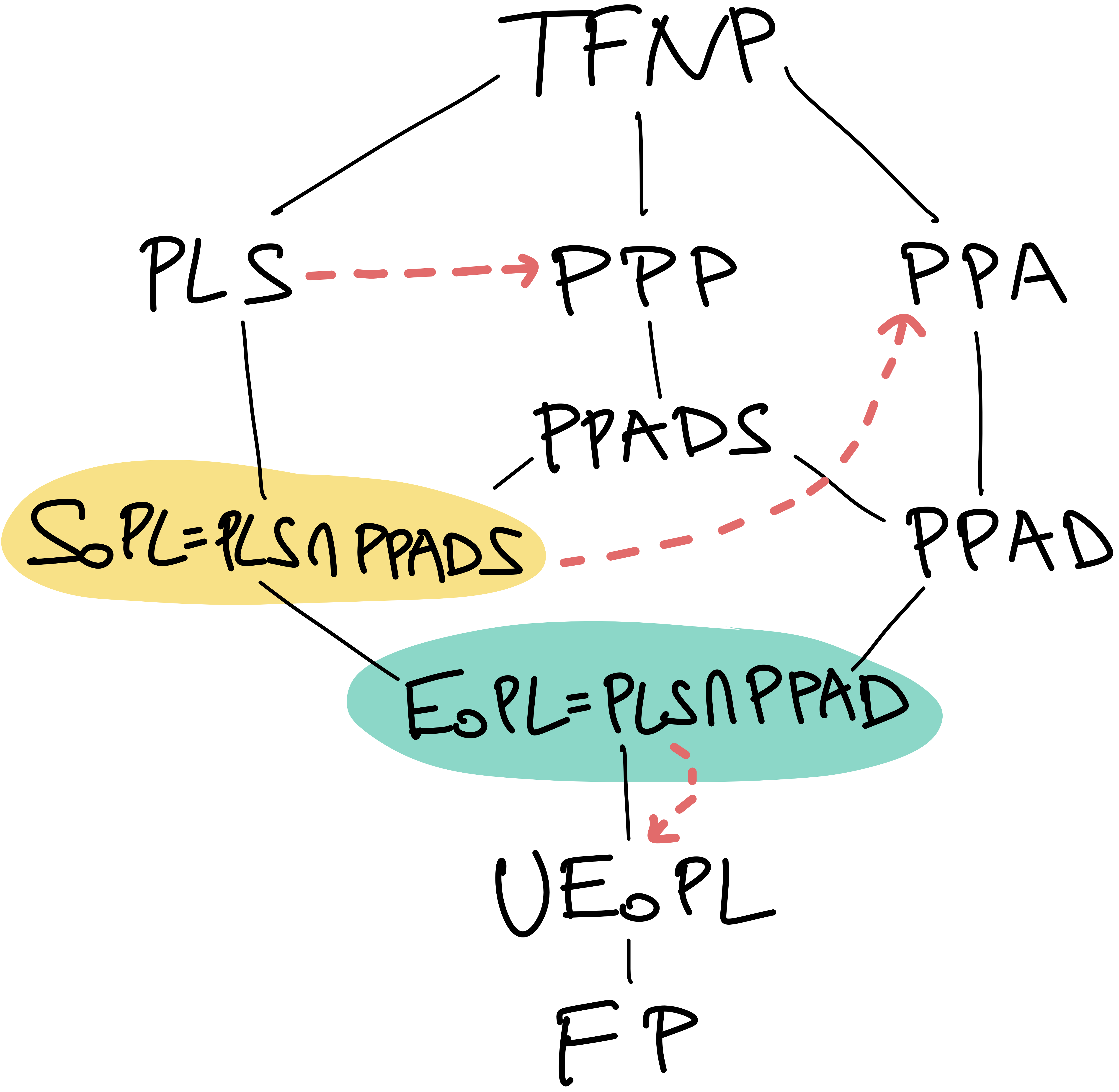
White-box sep. ⇒ P ≠ NP

Black-box sep. possible

Beame et al. 98'

Morioka 01'

Buresh-Openheim 04'



More Collapses?

NO MORE (BLACK-BOX)

OUR WORK

Image from https://fanart.tv/movie/9267/and-now-for-something-completely-different/

Resolution v.s. Sherali-Adams

Resolution

$$\frac{A \vee x \;,\; B \vee \neg x}{A \vee B}$$

measure : width

Simulated by

Sherali-Adams

$$\sum_i p_i(x) q_i(x) = 1 + J(x)$$

measure : degree

Resolution v.s. Sherali-Adams

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OUR RESULT: Simulation needs exp. large coefficients

Resolution v.s. Sherali-Adams

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measure: degree

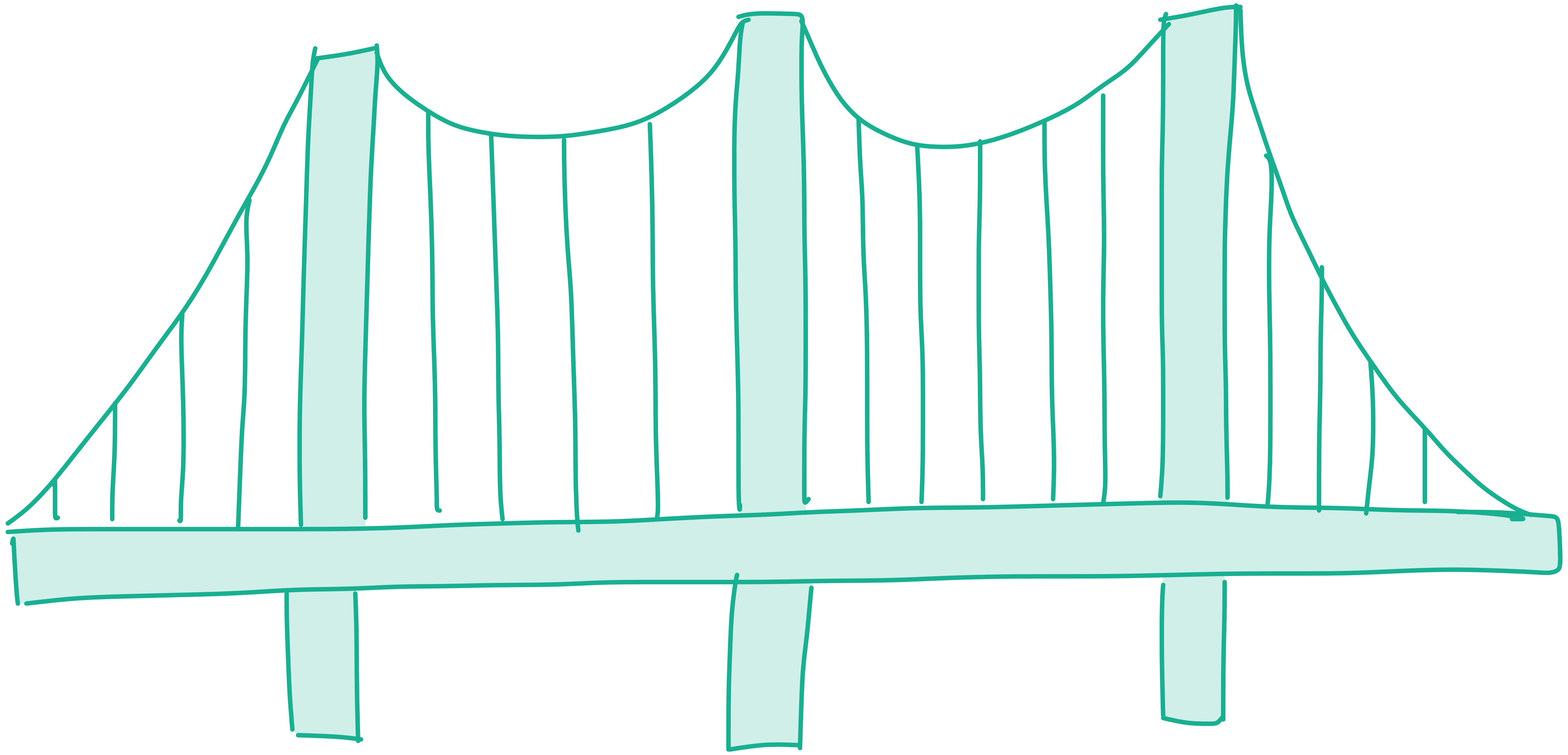
OUR RESULT: Simulation needs exp. large coefficients

$\Updownarrow$

PLS & PPADS

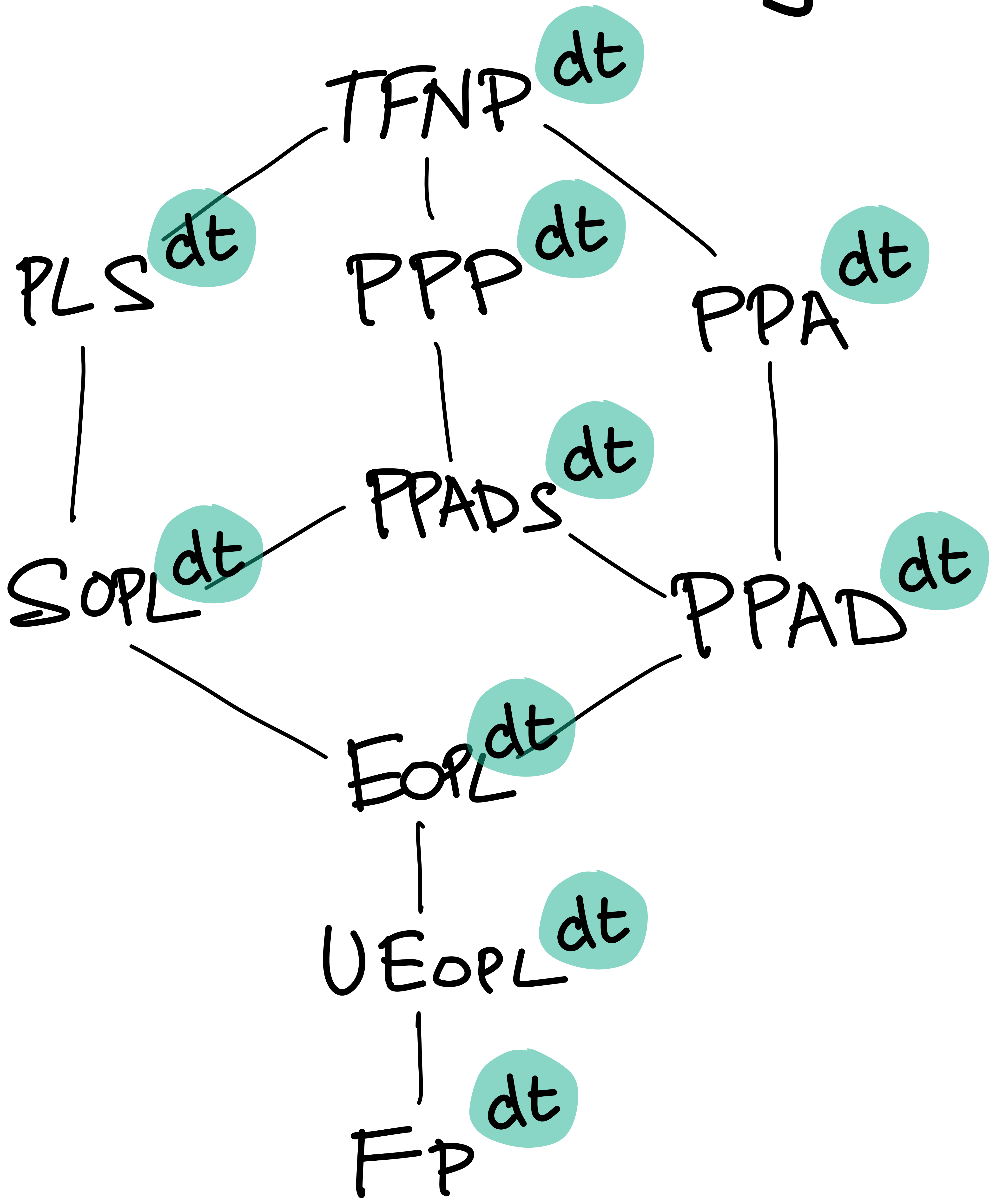
THE BRIDGE

Proof Complexity



TFNP

World 1: Query analogues



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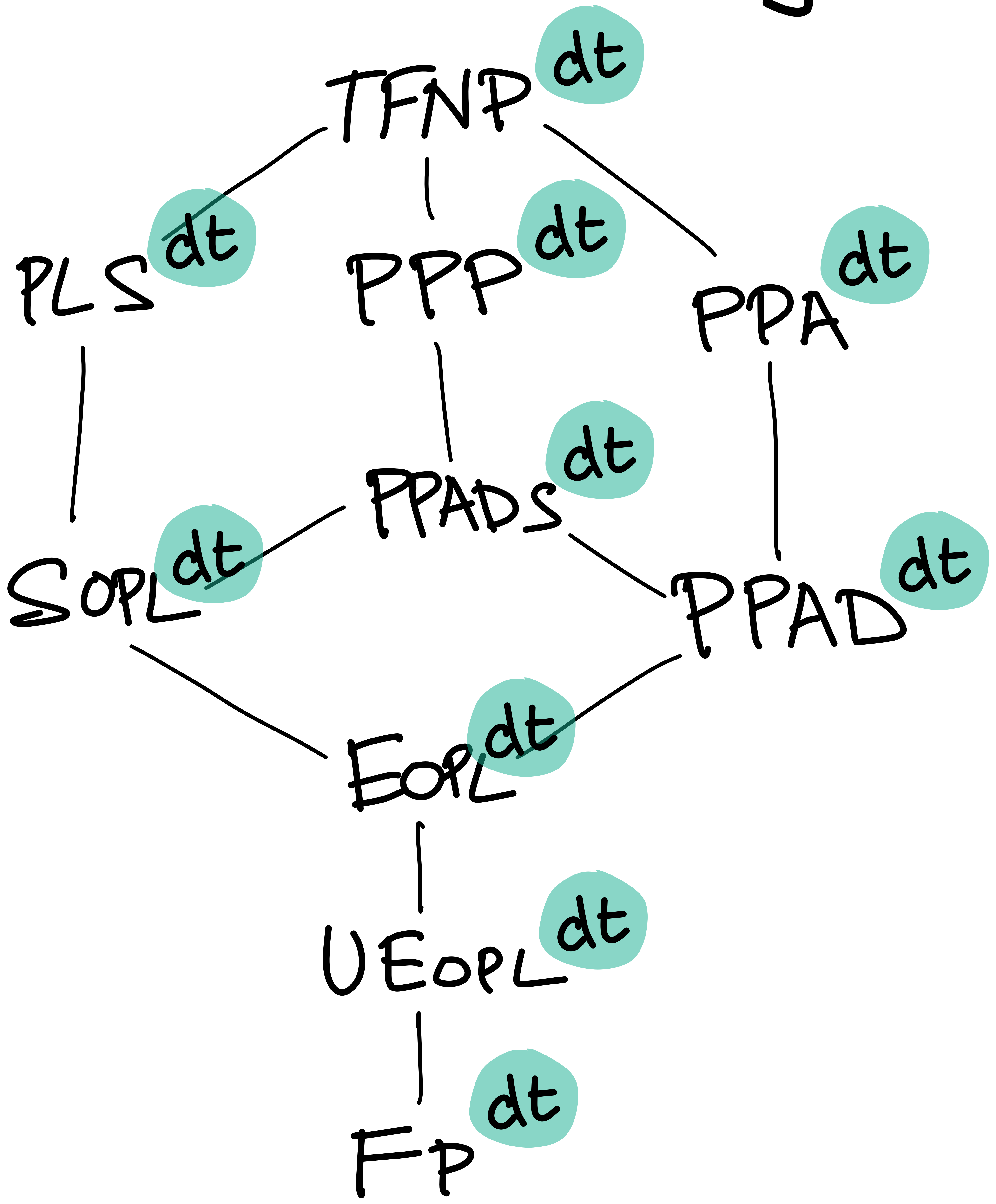
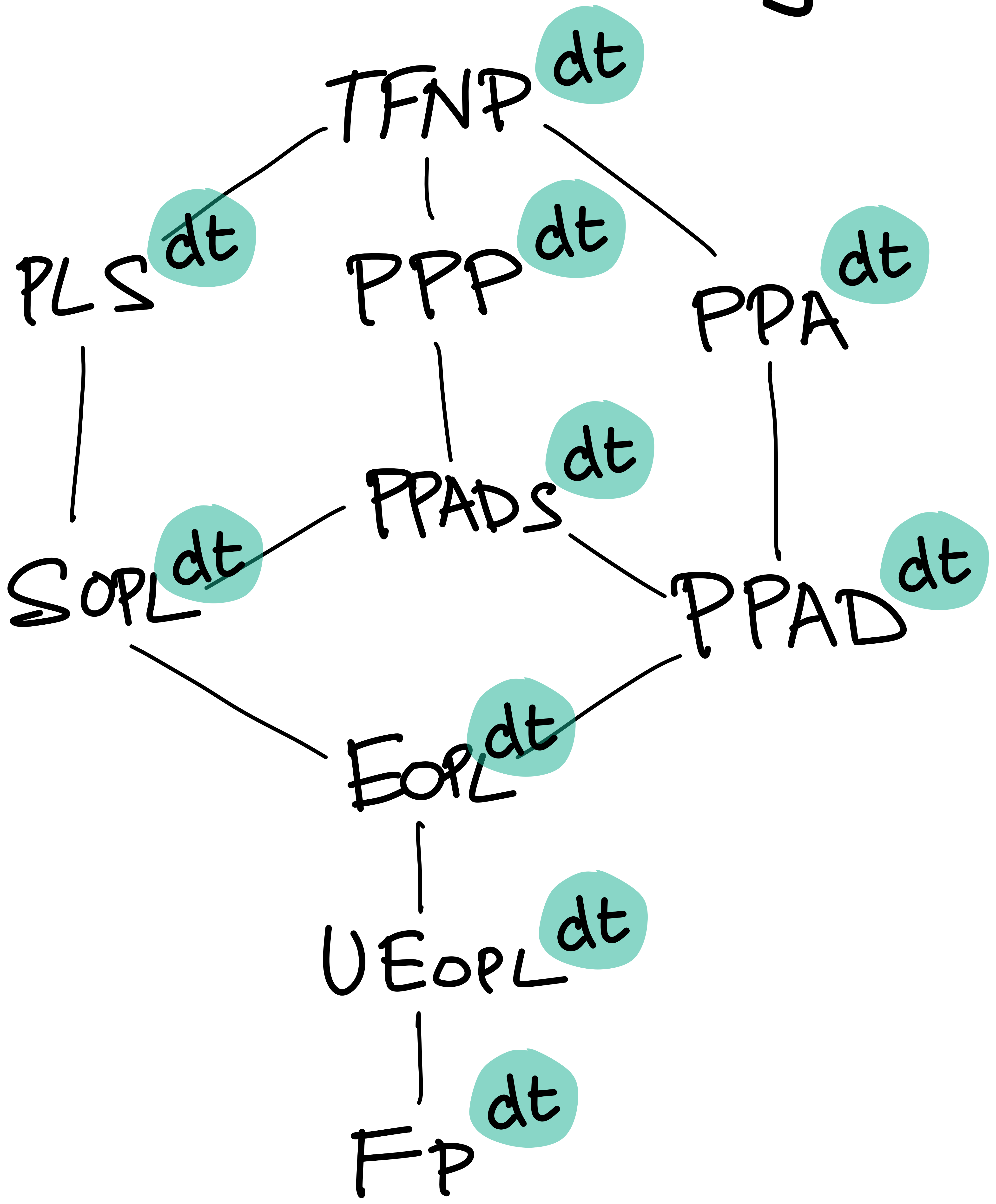


- dt
  $\equiv$
  query analogue

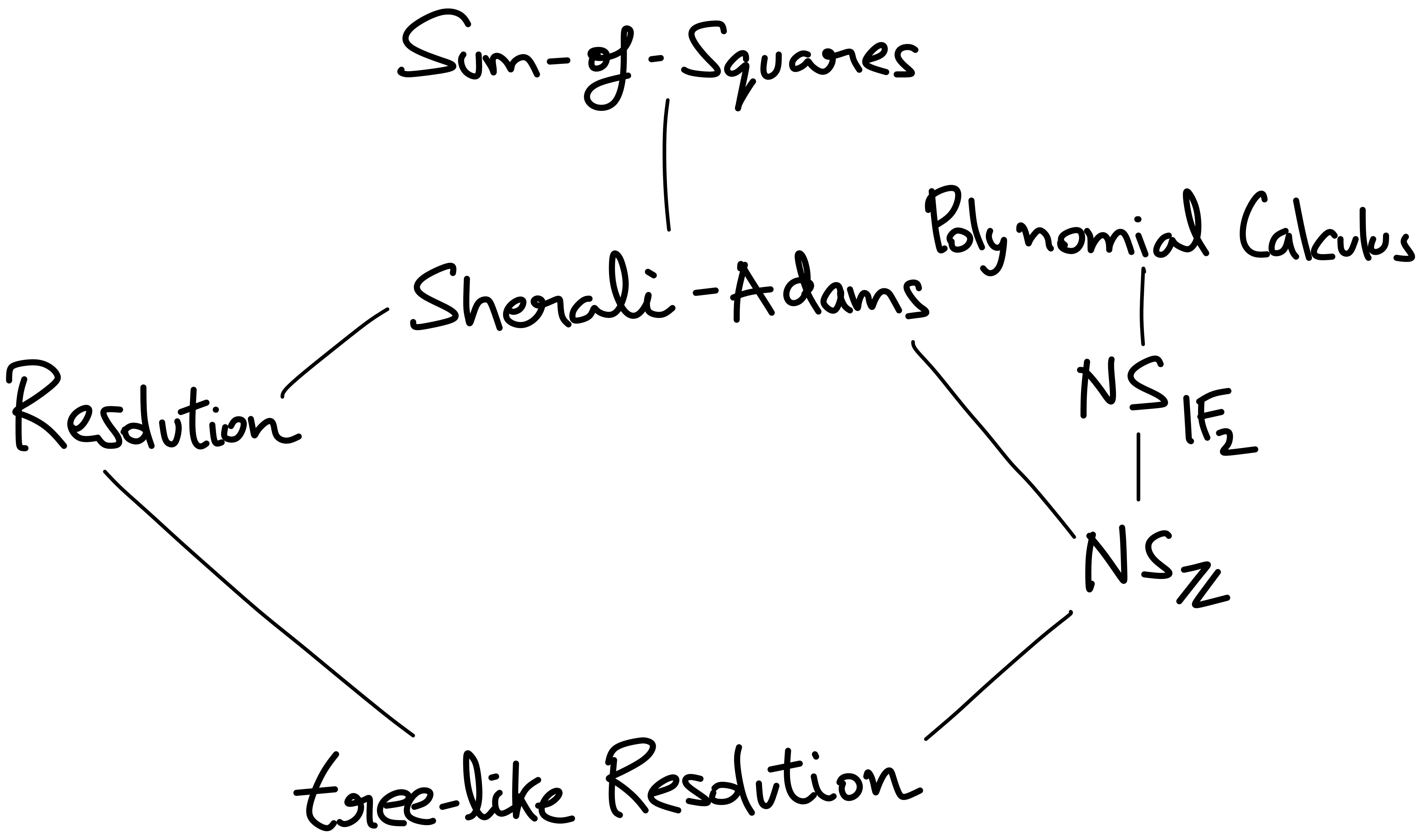
World 1: Query analogues



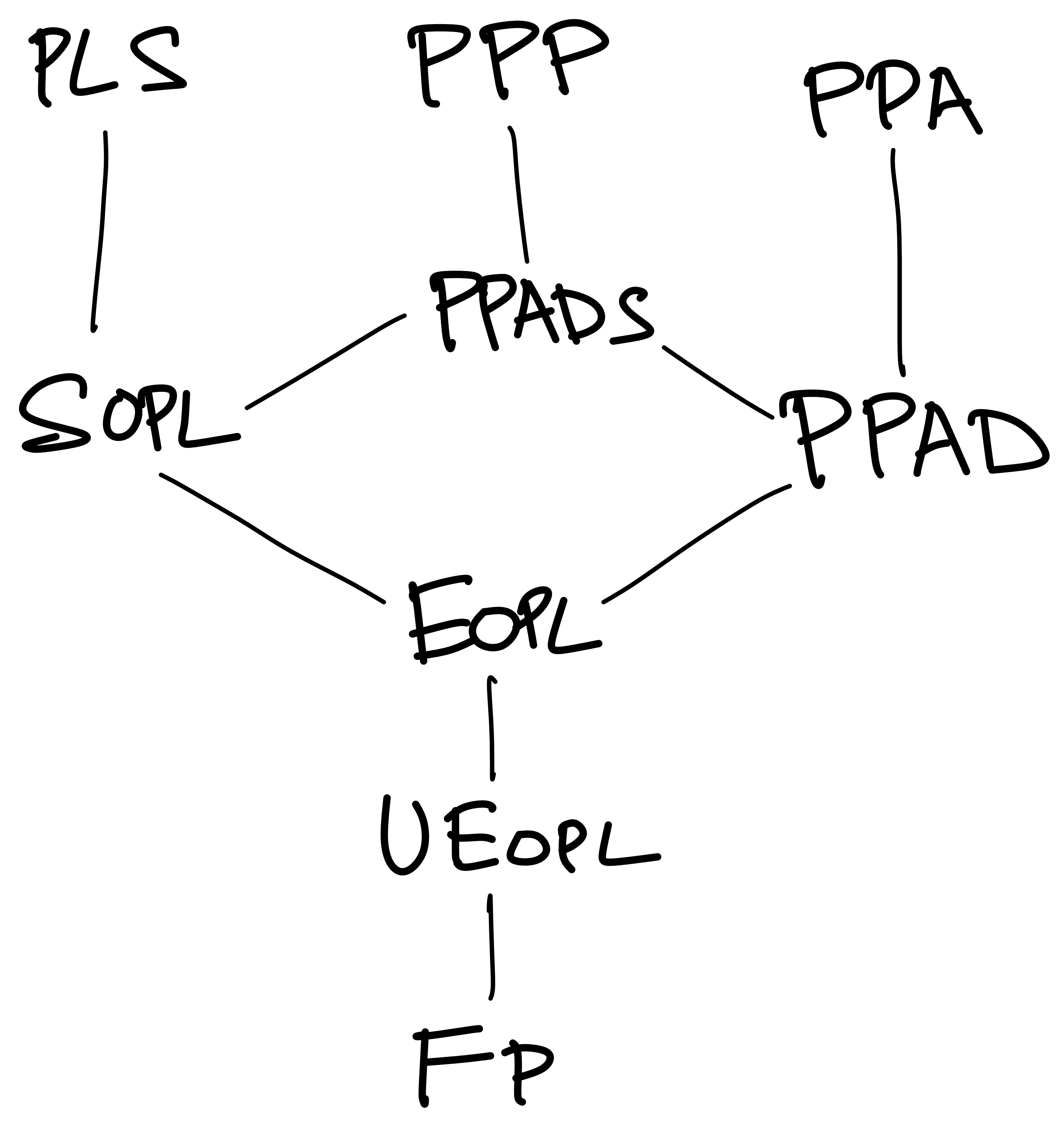
- dt ≡ query analogue
- Reductions ≡ shallow decision trees

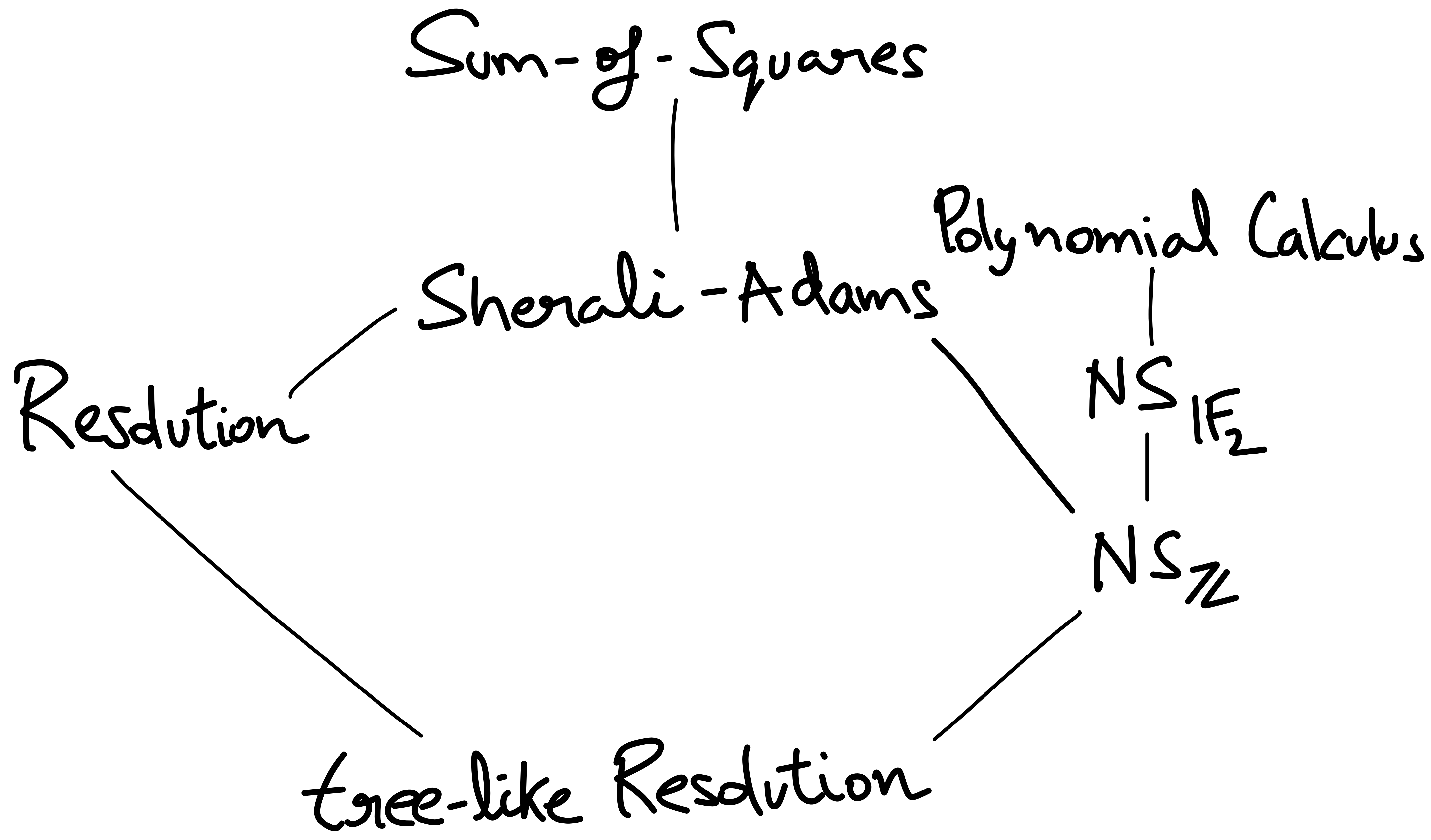
World 2: Proof Complexity

Is there a short derivation that this CNF is unsat?



Time to squint





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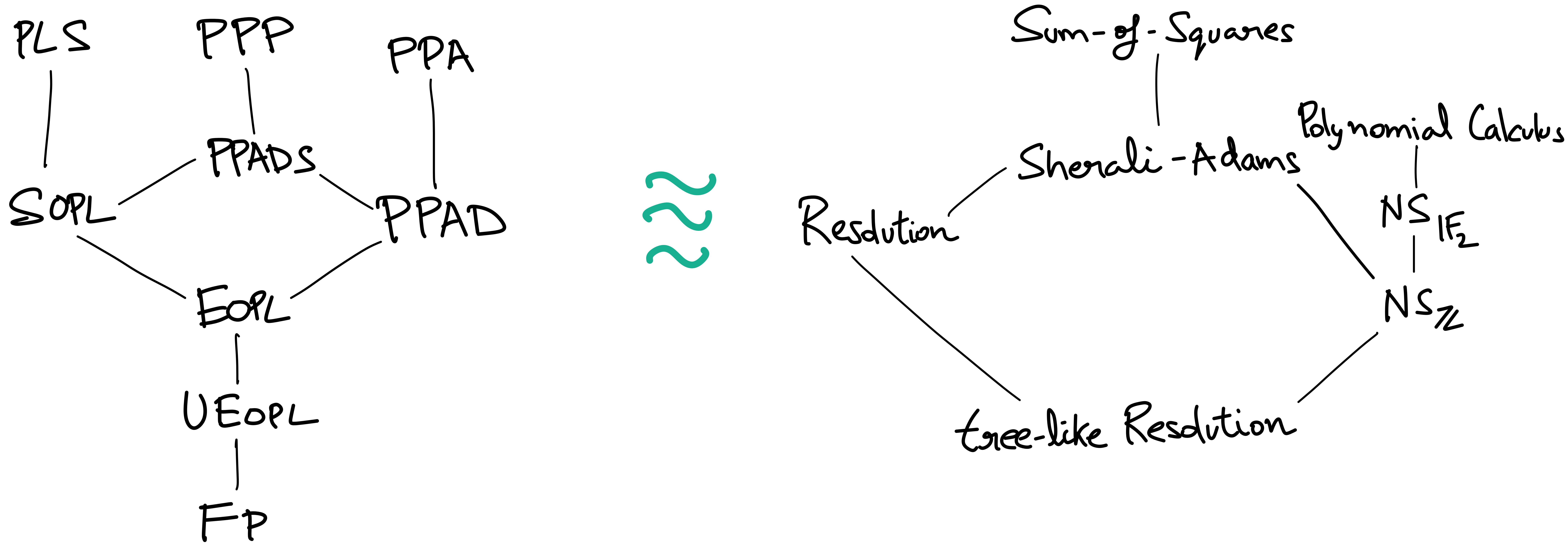

The Bridge: Characterizations

- TFNP^{dt} search problems can be translated into CNF fallacies

SINK-OF-DAG $\mapsto$ "this dag has no sinks"

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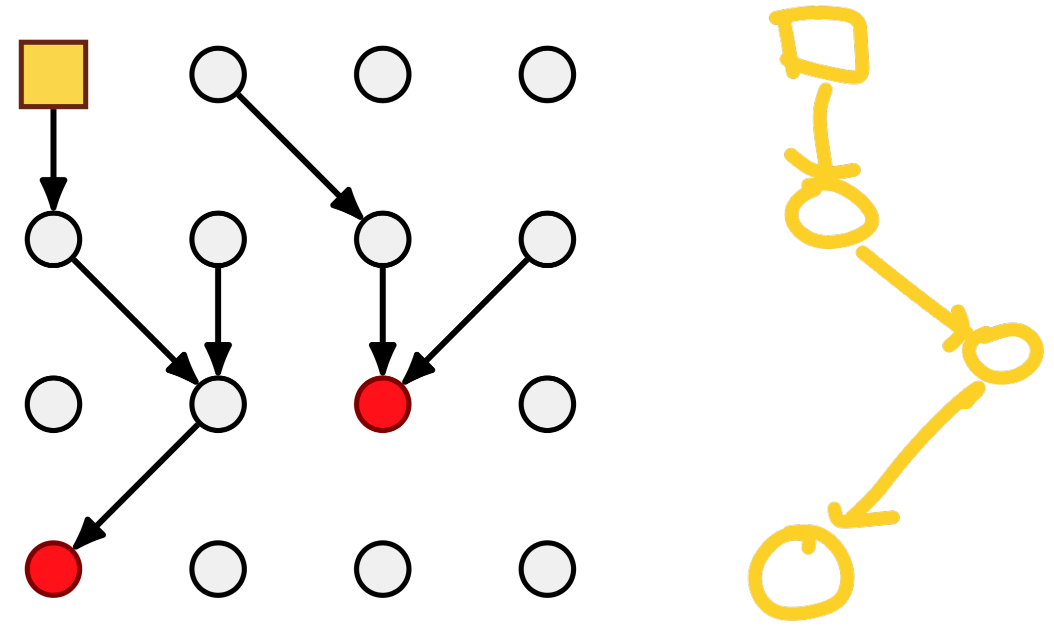
Example: Res Width $\lesssim \mathrm{PLS}^{dt}$ depth

Search $\mapsto$ CNF

Keep going down the dag



The Bridge: Characterizations

- TFNP^{dt} search problems can be translated into CNF fallacies

- CNF fallacies define search problems

$$\varphi = x_1 \wedge (\bar{x}_1 \vee \bar{x}_2) \wedge x_2 \longmapsto \text{find}(x_1, x_2)$$

falsified clause

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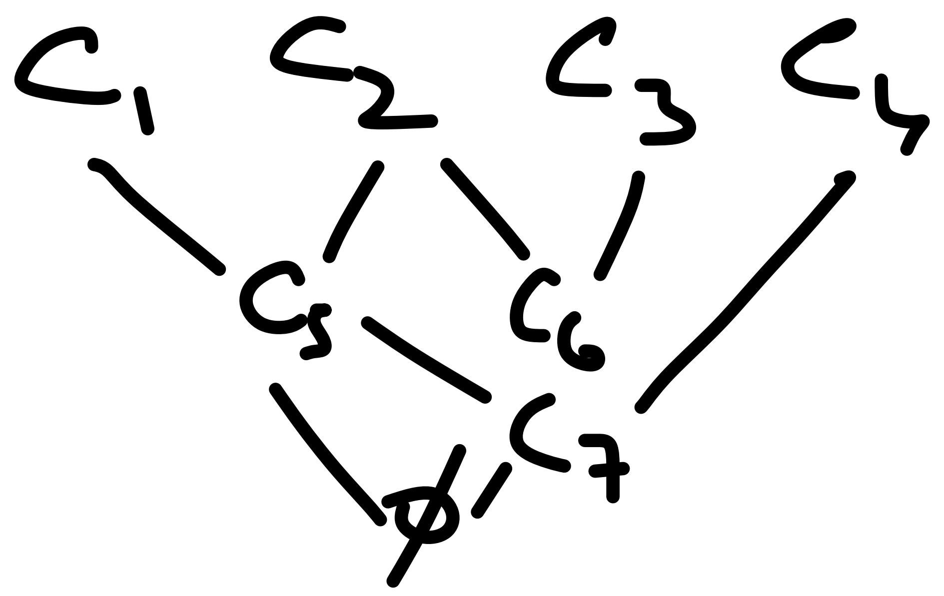
Example: Res Width $\gtrsim \mathrm{PLS}^{dt}$ depth

CNF $\mapsto$ Search

"Flip" proof



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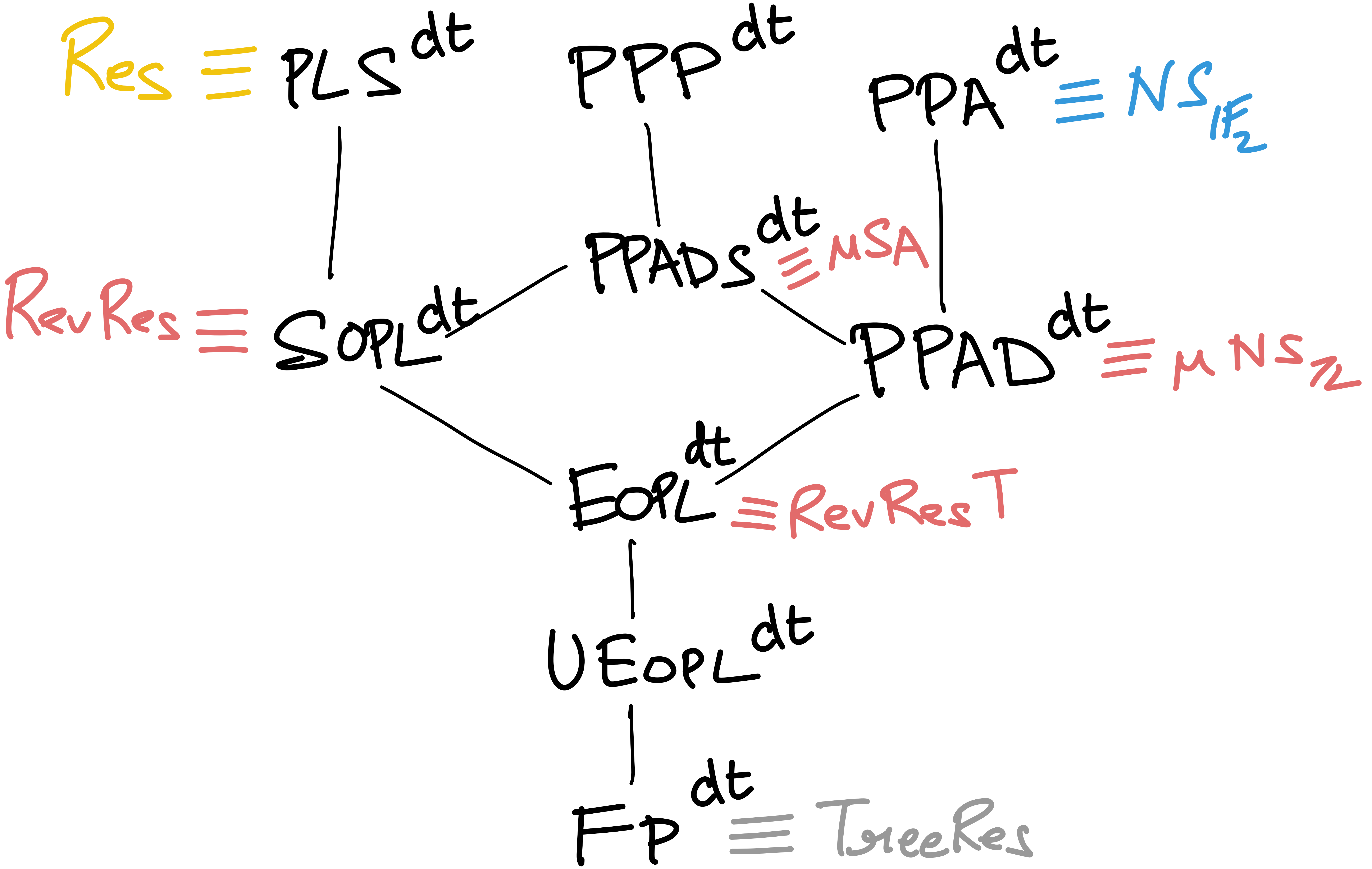

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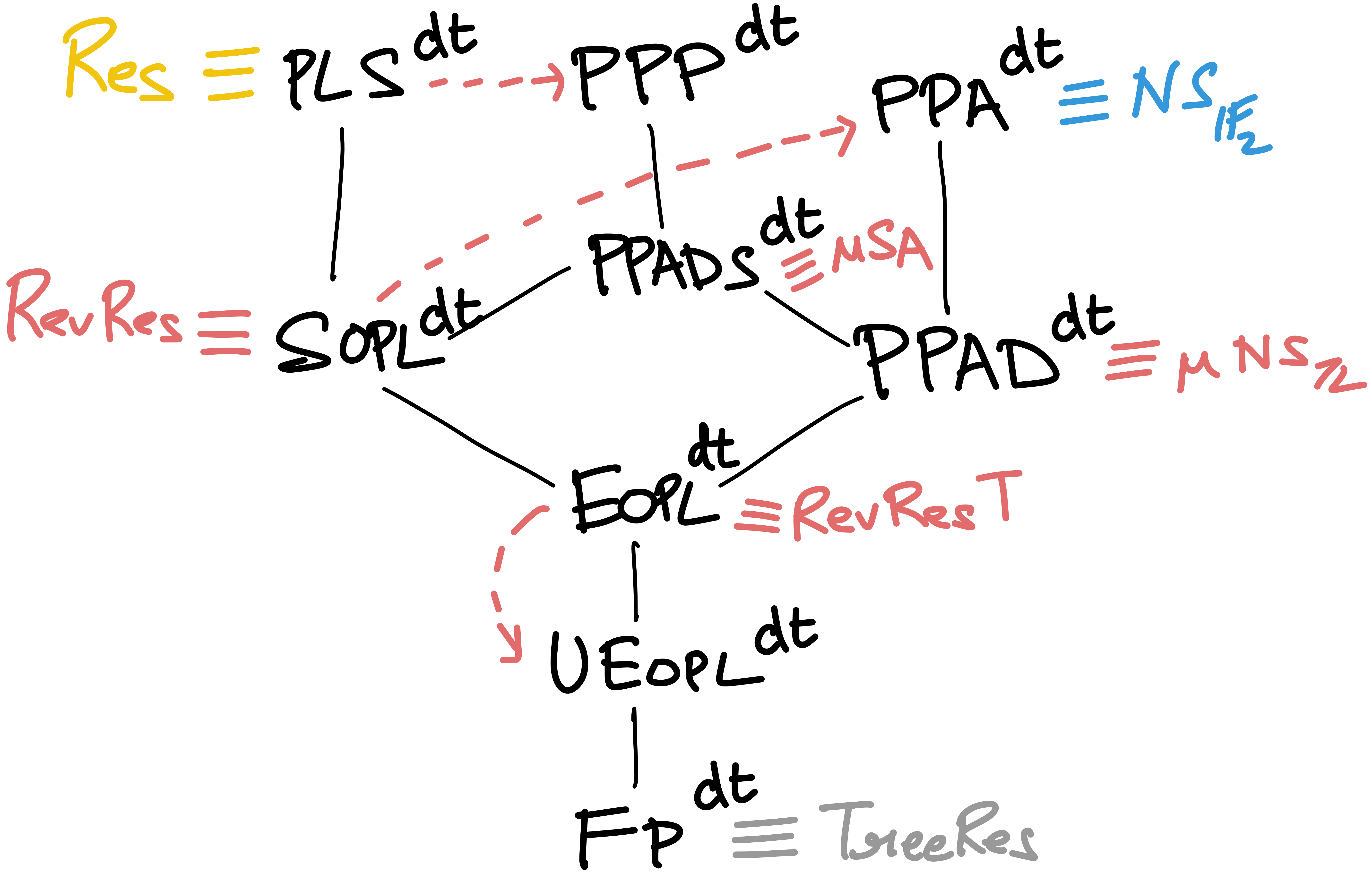

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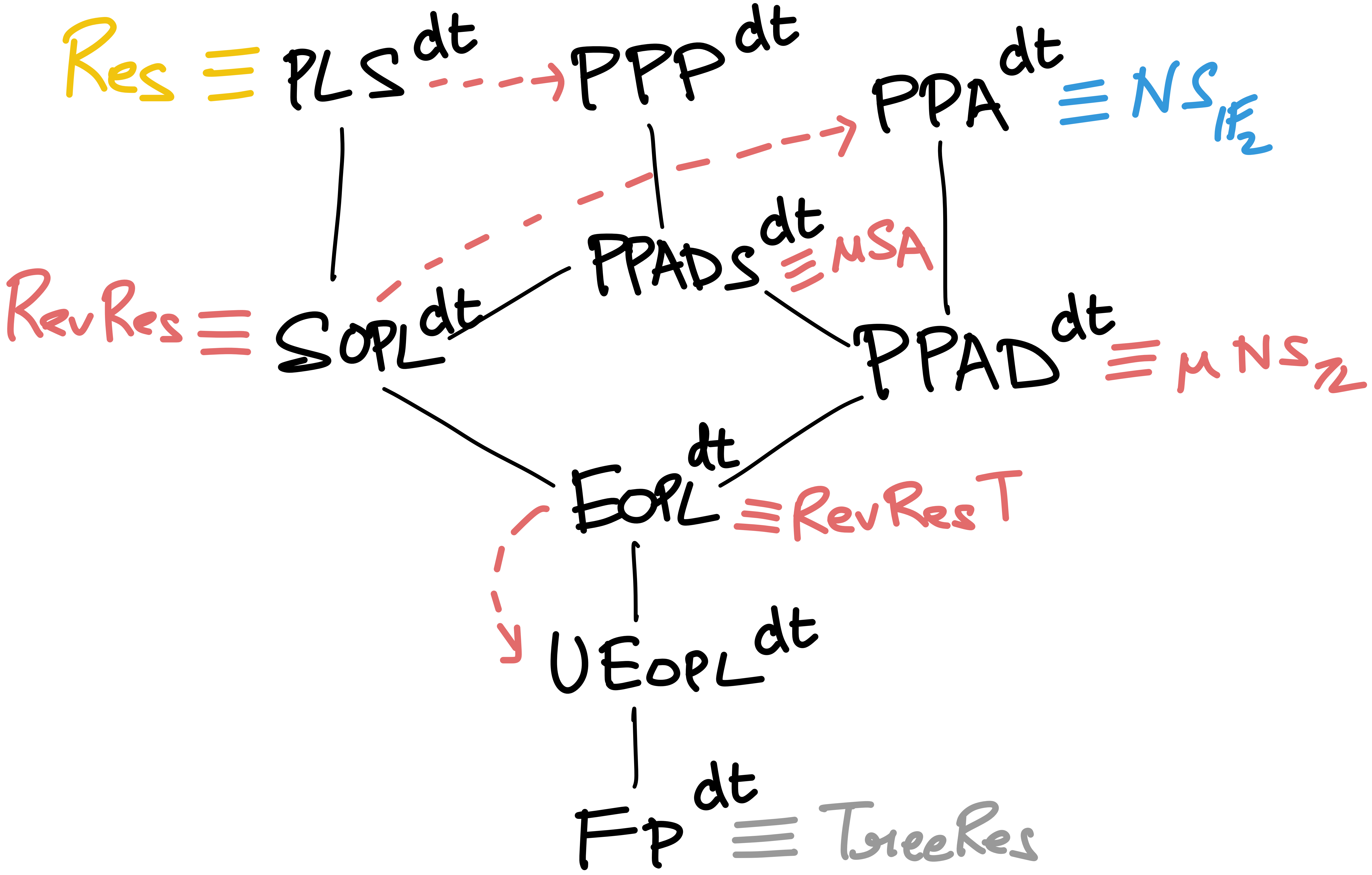
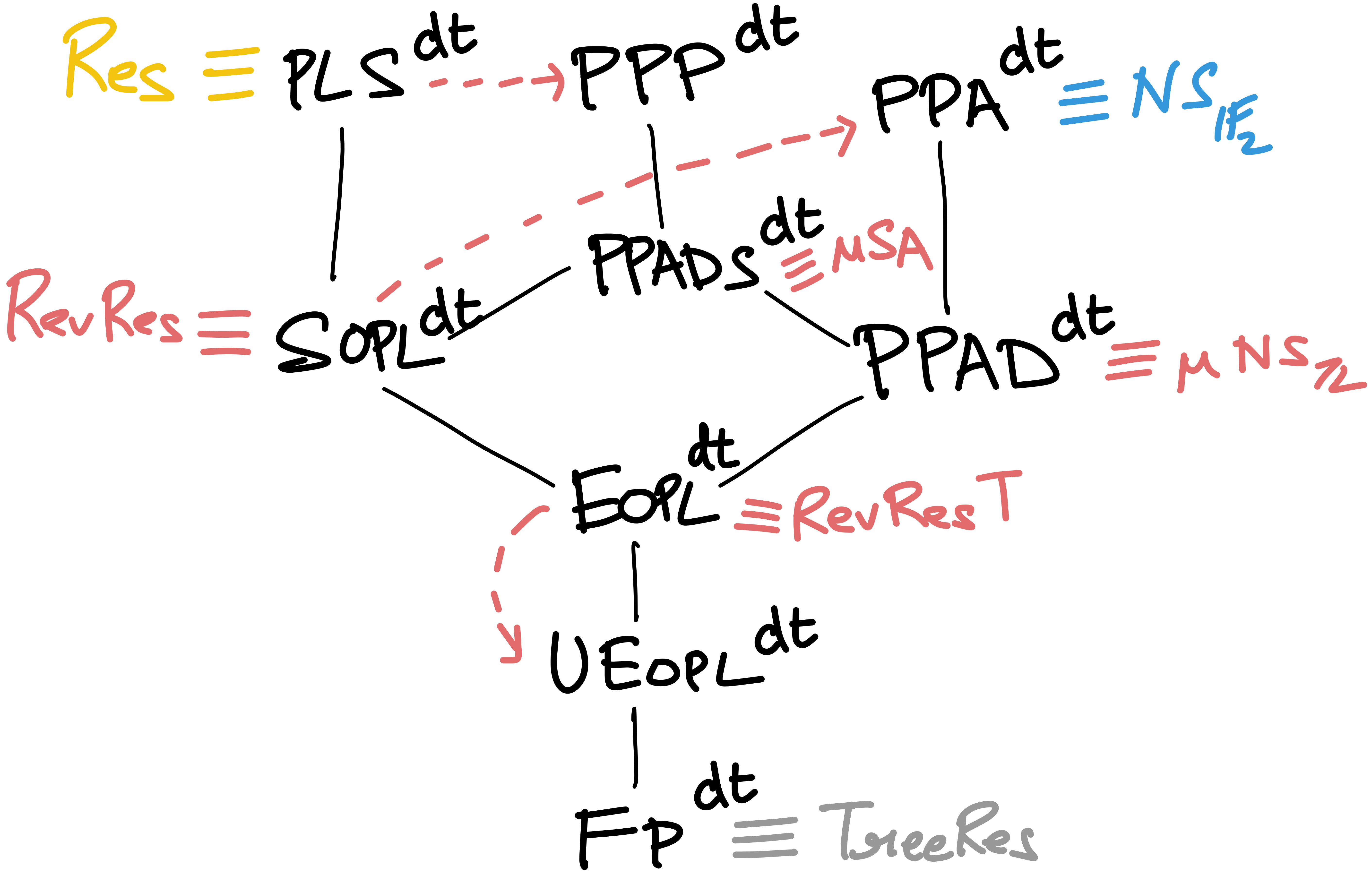


Results rephrased:

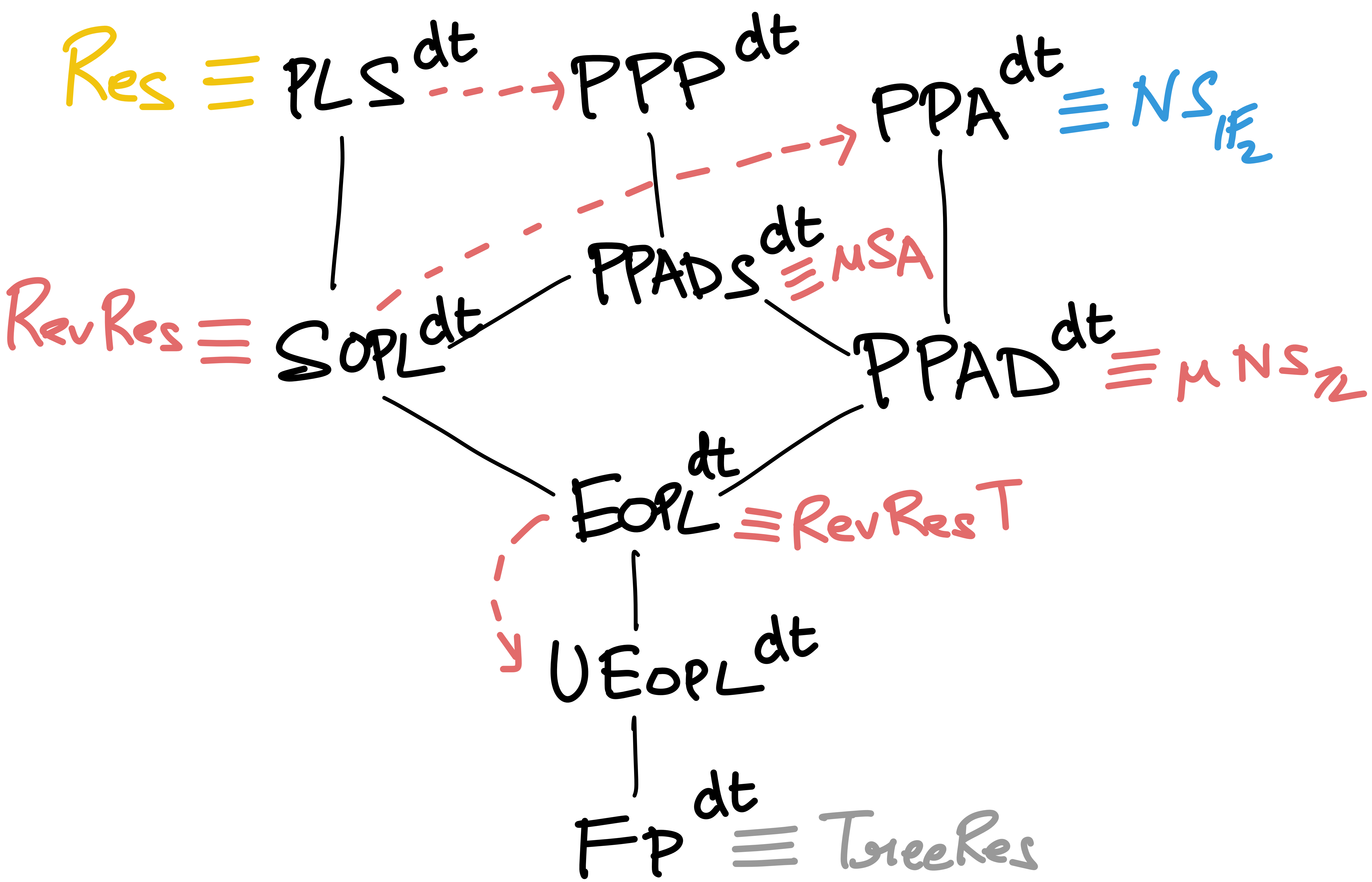
The Bridge: Characterizations



Results rephrased:

- $\mathrm{Res} \not\subseteq \mu\mathrm{SA}$

The Bridge: Characterizations



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- Res $\not\subseteq$ μSA
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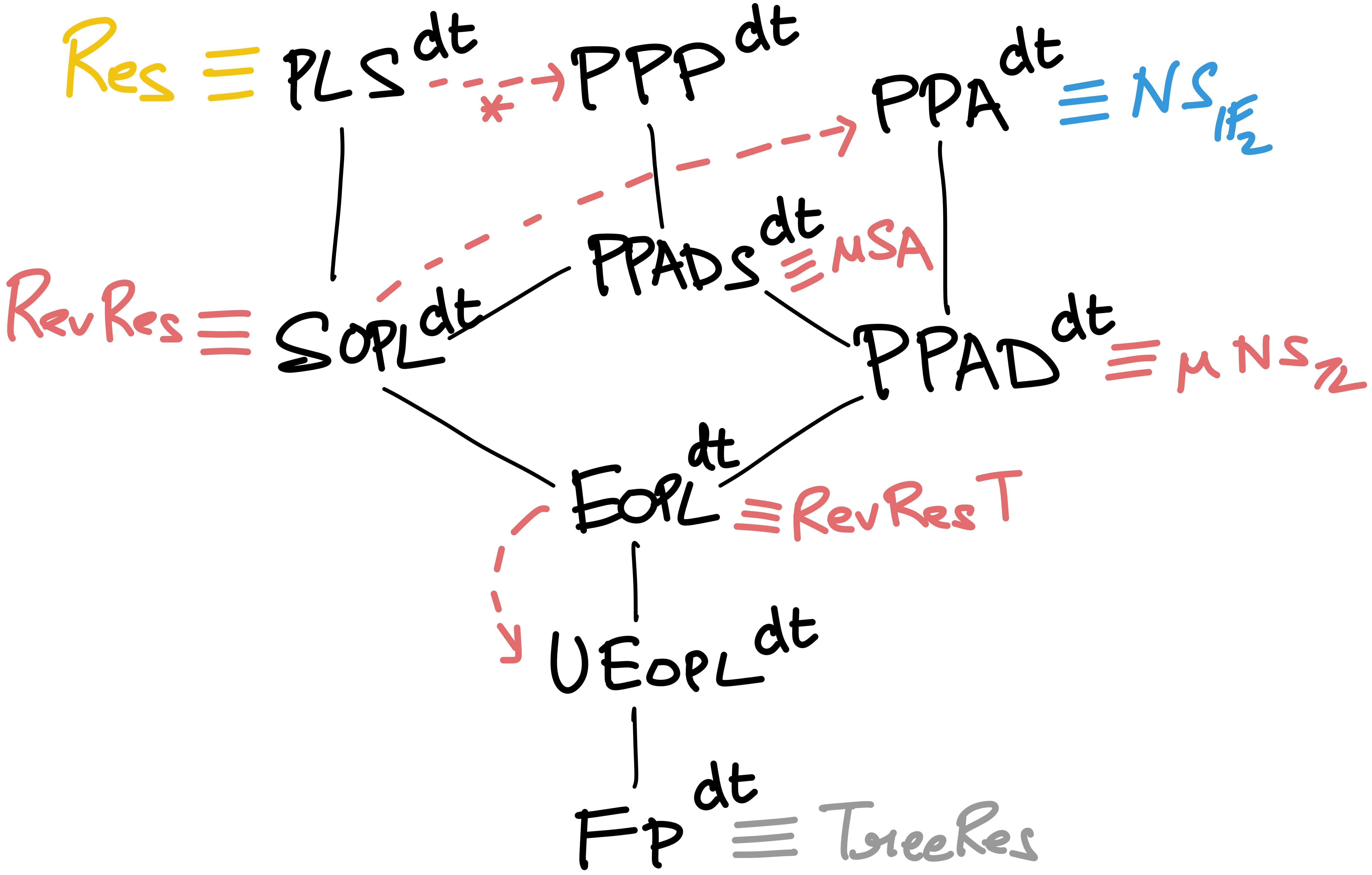
The Bridge : Characterizations



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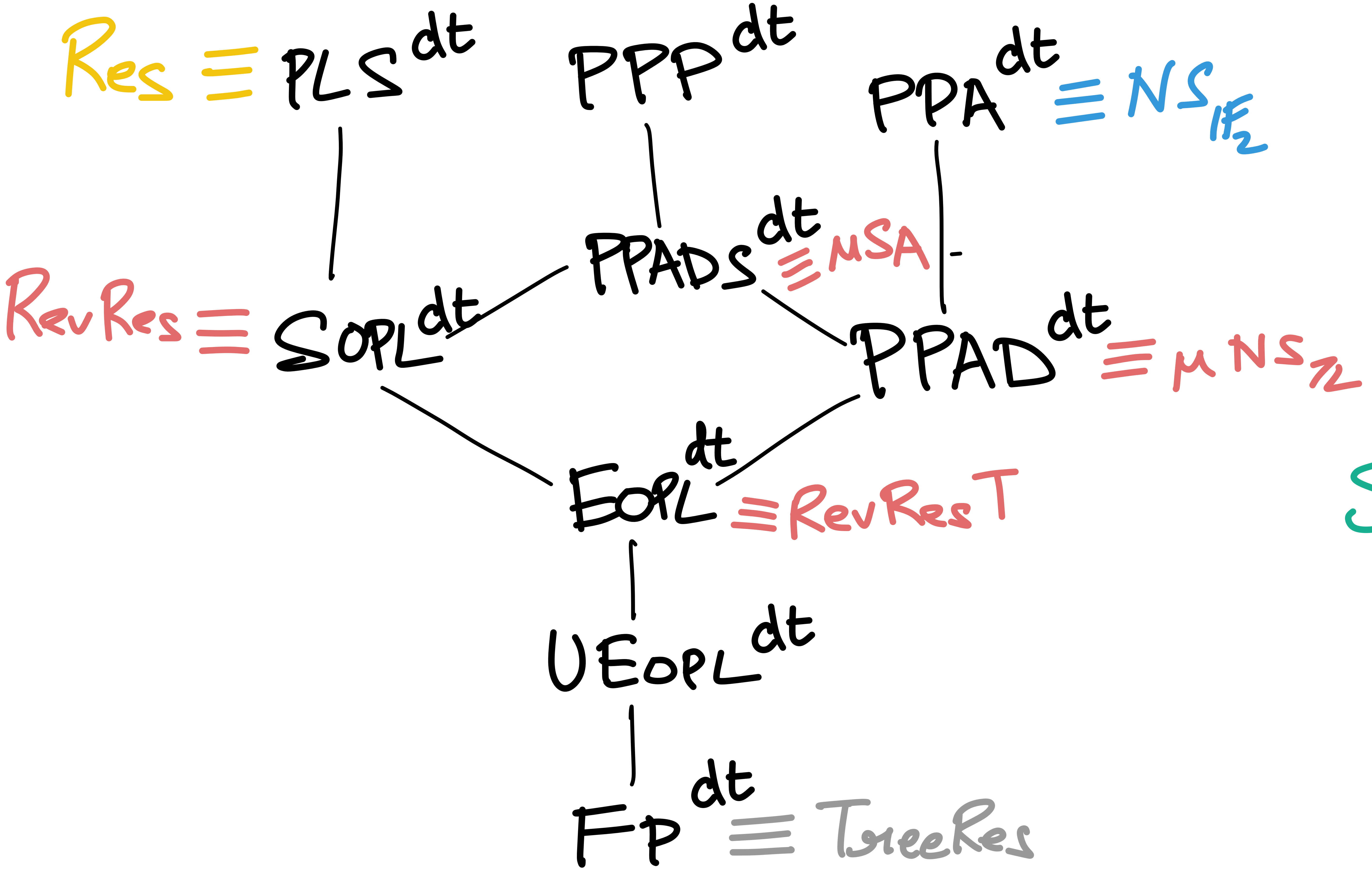
* Independent work [BT22]

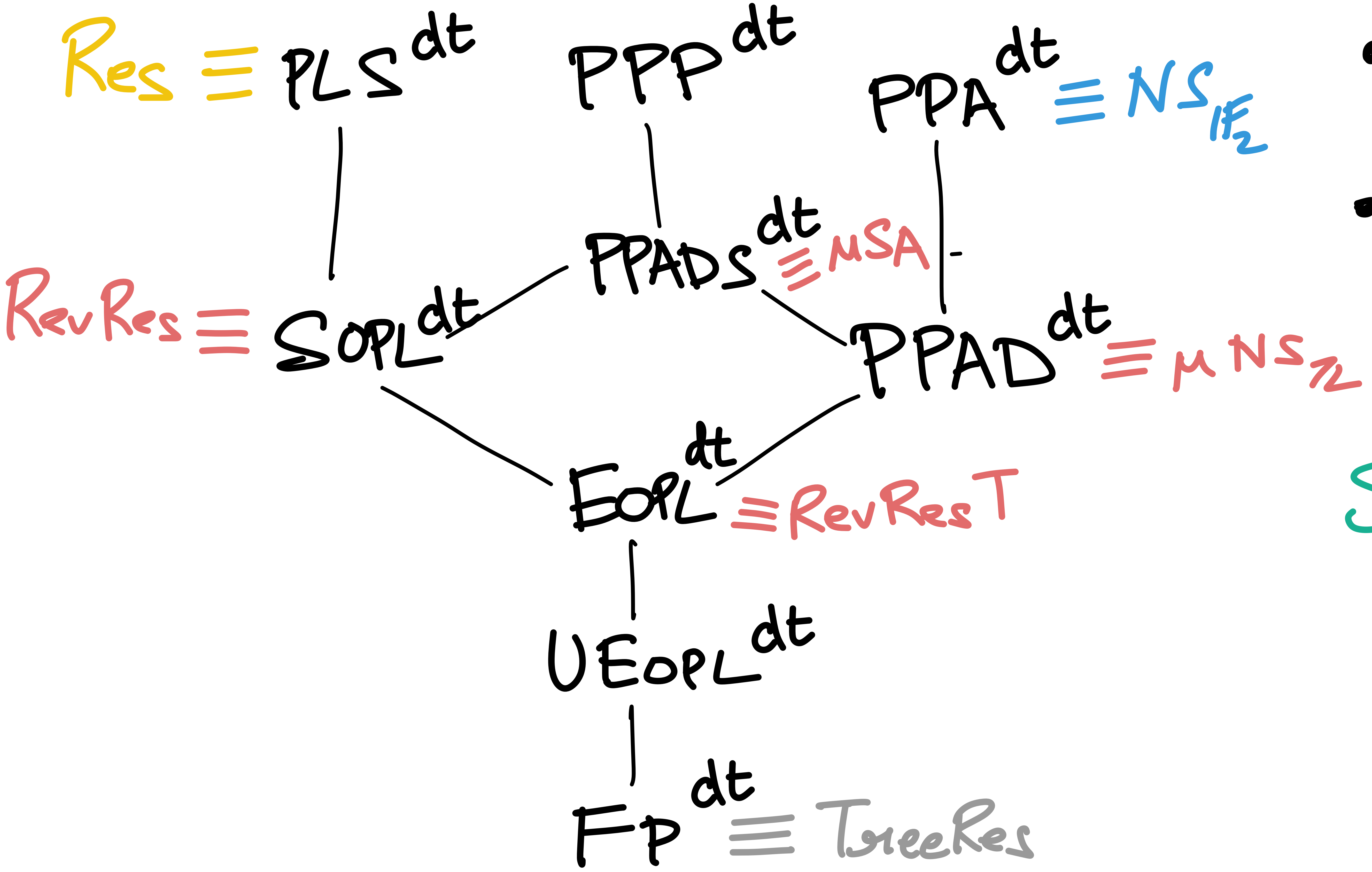
Open Problems



Characterizations

Structure of TFNP

Open Problems



Characterizations

- PPP? UEoPL?
- SoS? [BFI22]

~~Polynomial Calculus~~?

Structure of TFNP

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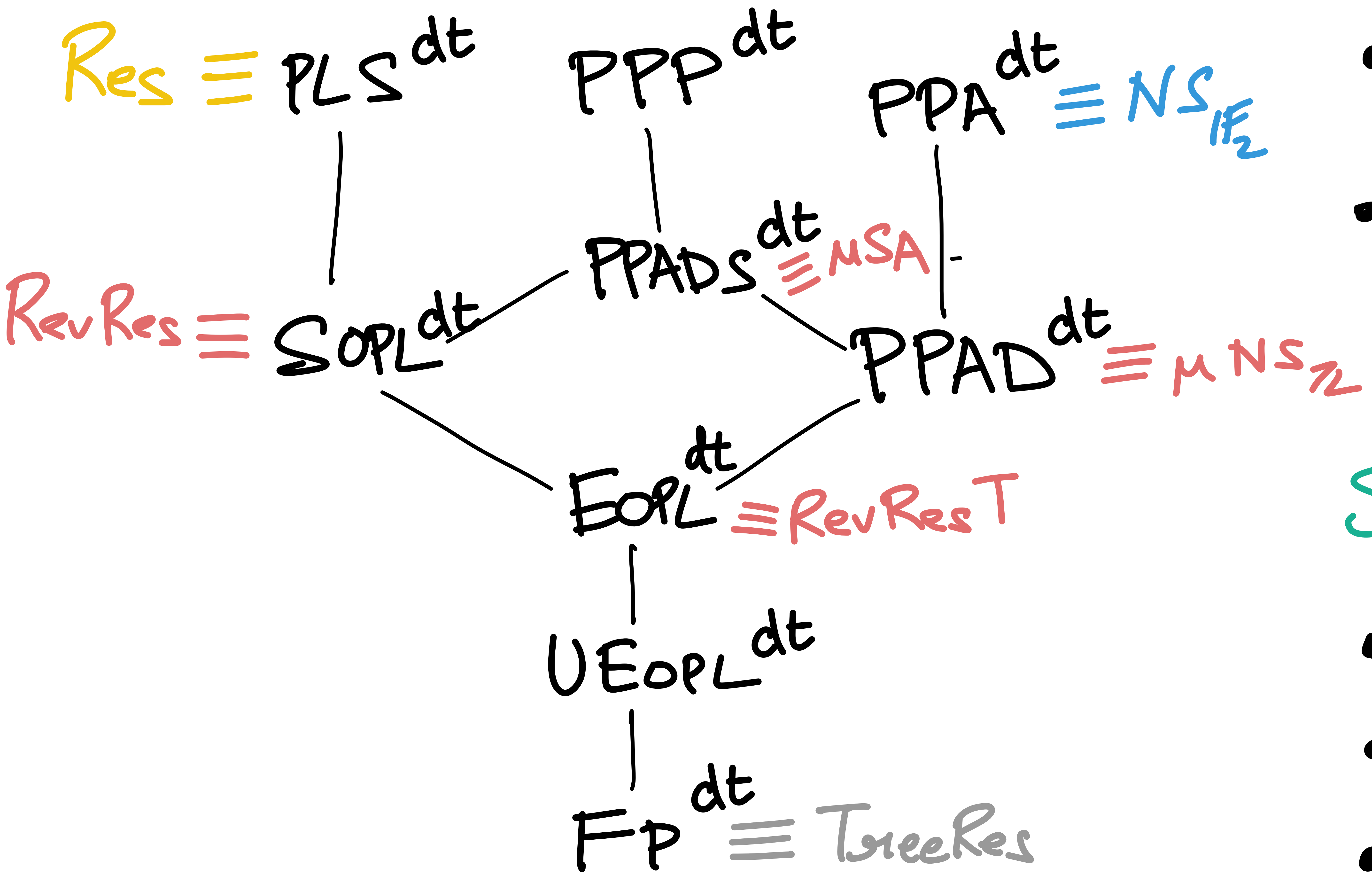

Characterizations

- PPP? UEOPL?
- SoS? [BFI22]
  ~~Polynomial Calculus~~?

Structure of TFNP

- PWPP?
- RAMSEY? SUNFLOWER?
- FACTORING?

Thanks!

for your attention

On Separations

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Key Lemma: Robust separation of SOPL from NS

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SoD without merging of paths

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$$\varepsilon\text{-NS} := \sum_{i \in [m]} p_i(x) \cdot a_i(x) = 1 \pm \varepsilon \quad \forall x \in \{0,1\}^n$$

On Separations

SoD without merging of paths

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NOTE: Not a Cook-Reckhow proof system!
Verification is CoNP-complete.

On Separations

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Lemma: Every $\frac{1}{2}$-NS refutation of SOPL_n requires deg $n^{\Omega(1)}$.

On Separations

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Lemma: Every $\frac{1}{2}$-NS refutation of $SoPL_n$ requires deg $n^{\Omega(1)}$.

IDEA: Randomized decision-to-search reduction in the style of Raz-Wigderson 92'.
We show that ε-NS proofs imply approx poly for OR.

On Separations

Lemma: Every $\frac{1}{2}$-NS refutation of SOPL_n requires deg $n^{\Omega(1)}$.

$\Downarrow$

Lemma: Any degree-$n^{O(1)}$ SA proof of SoD_{n^2} requires coefficients of magnitude $\exp(\Omega(n))$.